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SKILL ENHANCEMENT COURSE
SEM-III/IV/V/VI

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B.Sc. HONS/GEN

MATHEMATICAL LOGIC and SET THEORY

Introduction to Mathematical Logic
Propositional Logic ▪ Predicate Calculus
Set Theory ▪ Relation

Dr. Debjani Chakraborti
Dr. Nilangshu Acharya
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SKILL ENHANCEMENT COURSE

MATHEMATICAL LOGIC and SET THEORY

A Complete Text Book

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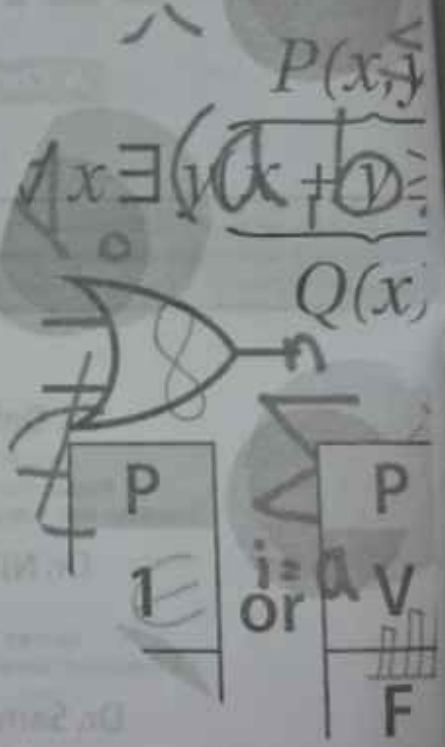

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Dedic
My maternal grandparents
My paternal grandparents La
My mother in law Late Kalyan
My parents Mrs. A
My
My daughter Deban
M
M
My son S

Dedicated To Our Beloved Family Members

My maternal grandparents Late Amiya Prova Ganguly and Late Santi Ranjan Ganguly

My paternal grandparents Late Bharat Chandra Chakrabarti and Late Labanya Chakrabarti

My mother in law Late Kalyani Bhattacharjee and father in law Late Byamkesh Bhattacharjee

My parents Mrs. Arati Chakraborty and Mr. Nani Bhushan Chakrabarti

My husband Dr. Bibhutosh Bhattacharjee

My daughter Debanjana Bhattacharya and son Debopratim Bhattacharya

and

My Uncle Late Binoy Bhushan Acharjee

and

My father Late Pankaj Kumar Bhandari

My mother Mrs. Malati Bhandari

My wife Soma Bhandari

My son Sagnik Bhandari and daughter Shrinika Bhandari

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01 Introductio Mathematical

Introduction

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It provides rules and techniques by which we ca
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2. All dogs like bones. Diki is a dog. He

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01 Introduction to Mathematical Logic

Introduction

Logic is a discipline that deals with the methods of mathematical reasoning. It provides rules and techniques by which we can determine whether a particular reasoning or argument is valid. Logic is a study of a chain of reasoning, and consistency of assumptions. In Logic we are not concerned with the contents of an argument, rather we are concerned only with the form of an argument. While reasoning about any particular fact, we construct arguments to support our conclusions and justify our beliefs. However, not all reasons are good reasons. Therefore while confronting an argument, we always ask a basic question :

"Does the assumed presimes give rise to the obtained conclusions?"

To answer this question there are objective criteria; in the study of logic we seek to discover and apply those criteria. We may distinguish between good and bad reasoning by using the methods and techniques of logic.

For example, consider the two arguments :

1. All men are mortaL Socrates is a man. Hence, Socrates is mortal.
2. All dogs like bones. Diki is a dog. Hence, Diki likes bones.

Both have the same form : All P are Q. Z is an P. Hence, Z is a Q. The truth or falsity of the particular premise *P* and conclusion Q is of no concern to logicians. They are only concerned with the fact that whether the premisses imply the conclusion. The main task of the logicians is systematic formalization of valid methods of reasoning. If the frame of analysis uses mathematical operations or techniques or if it is primalily focussed to the study of mathematical reasoning, then it is called mathematical logic. We can understant the domain of mathematical logic if we define its principal aim to be a precise and adequate understanding of the notion of mathematical proof.

Impeccable definitions have little value at the beginning of the study of a subject. The best way to find out what mathematical logic is about is to start devoting into a deep meaning and purpose of the study of the subject. Although

logic is basic to all other studies, its fundamental and apparently self-evi character discouraged any deep logical investigations until the late 19th cen Then, under the impetus of the discovery of non Euclidean geometry and desire to provide a rigorous foundation for calculus and higher analysis, inte in logic revived. This new interest, however, was still rather unenthusiastic un around the turn of the century, the mathematical world was shocked by discovery of the paradoxes - that is, arguments that lead to contradictions.

Greek philosopher, Aristotle (384–322 BCE), was the pioneer of logi reasoning. Logical reasoning is used in many disciplines to provide valid resu It provides the theoretical base for many areas of mathematics and consequen computer science. The proofs of theorems in mathematics can be established Logical reasoning. It also has many practical applications in computer scien like design of computing machines, artificial intelligence, verification of correctness of computer programs, definition of data structures for programmi languages etc.

Logic is the basis on which all the sentences are built. The mathematical theor of logic is called symbolic logic. This algebraic way of studying the arguments wa developed by the English Mathematician George Boole in the19th century.

This chapter deals with logical symbols which are used to state and appl rules of valid inference and hence construct correct mathematical arguments.

The basic building block of logic is proposition. Let us have an indepth analysi of the concept of propositions.

1.1 Propositions or Statements

In logic we construct and evaluate arguments; arguments are built with propositions. Although these concepts are apparently simple, they require careful analysis.

A declarative sentence (or assertion) which is either true or false, but not both, is called a proposition (or statement). A Statement is a proposition that is typically asserted by a declarative sentence, but not the sentence itself. There are some logicians who prefer to use statement in place of proposition, although the latter has been more commonly used in the history of logic. Every statement must be either true or false, although the truth or falsity of a given statement may be unknown.The sentences which are interrogative, exclamatory or imperative in nature are not propositions. Since a question, a command or order and an exclamation asserts nothing, and therefore they are not propositions. Propositions are denoted by the letters p, q, r etc. If the proposition is true we give it a truth value T (or 1) otherwise F (or 0). Propositions are the building blocks of our reasoning.

Example 1.1 : For example let us consider the following sentences :

1. p : "It is raining today". This statement is a declared statement and so it is a proposition. It may be true or false.
2. q : "What is your name?" This statement is an interrogative statement and not declarative in nature as it is neither true nor false. So, it is not a proposition.
3. r : "$a + b = 3$". This statement is not a proposition since it is not true if $a = 1$, $b = 4$ and it is not false if $a = 1.5$, $b = 1.5$.
4. p : "Kolkata is the capital of West Bengal". This statement is a proposition because it is a declared statement which is true.
5. q : "Take a cup of tea". This is a command statement and is not a declarative statement. So, it is not a proposition.
6. r : "$2 + 2 = 3$". This statement is a proposition which is false.
7. p : "Oh my gosh!". This is an exclamatory statement and so it is not a proposition.

When we state some proposition, we do so using a sentence in a particular language. However, the proposition we propound is not identical to that sentence. This is evident because two different sentences, consisting of different words differently arranged, may have the same meaning and may be used to declare the very same proposition. For example, "India won the cricket match" and "the cricket match was won by India" are plainly two different sentences that make the same assertion. Sentences are always parts of some language, but propositions are not tied to English or to any given language. Let us consider the following sentences :

Today is Sunday (English Language)

আজ রবিবার (Bengali Language)

आज रविवार (Hindi Language)

are in different languages, but they have a single meaning : all three, using different words, may be used to assert the very same proposition. Thus, proposition is the term used to refer to what it is that declarative sentences are typically used to assert.

Propositions may be simple, but they may also be compound, If the truth value of a given statement p can be obtained out of p only and does not rely on any other statement, then p is called simple proposition or an atom. A statement obtained by combining two or more propositions by certain connecting words (mathematically called logical operators) we obtain a new statement called compound proposition. Consider the following propositions :

p : t_1, t_2 are integers and q : $t_1 + t_2$ is an integer ($t_1 + t_2$ is the sum of t_1 and t_2)

Combining the two propositions p and q, we get a compound proposition

r : If t_1, t_2 are integers then $t_1 + t_2$ is also an integer.

Notes

1. A mathematical paradox is a statement which contradicts itself b aparantly seems to be logical. So a paradox is that which is neither tru false. For example "I am speaking the truth". This statement is a parad there is a doubt about the truth or falsity of the statement.
2. *Argument* is a technical term in logic. An *Argument* is a group of proposi of which one is asserted to follow from the others along with only conclusion.
3. In an argument, the *Premises* are the propositions upon which inferenc build; the propositions that are asserted to provide the basis or reasons the conclusion.
4. *Inference* is a process by which one proposition is arrived at and asserted the basis of some other proposition or propositions.
5. In any argument, the *Conclusion* is the proposition to which the oth propositions in the argument are asserted to give support, or for which th are given as reasons in order to convince the reader.

1.2 Forms of Logic

An argument consists of a finite sequence of sentences, called premi together with another sentence, the conclusion, which the premises are tak to support. There are two very different ways in which a conclusion may reinforced by its premises, and thus there are two forms of arguments : t *deductive argument* and the *inductive argument*. Understanding this distinct is important in the study of logic.

A deductive argument makes the claim that its conclusion is supported its premises determinatively with complete certainty. An inductive argument contrast, does not make such an assertion. An inductive argument asserts that premises give only some degree of probability, but not certainty, to its conclusi Using deductive argument it is possible to arrive at a correct logical argument b Inductive argument may not provide a correct conclusion. For example we m explain deductive argument as :

Premises : All squares are rectangles. All rectangles have four sides

Conclusion : All squares have four sides

Inductive argument usually involves a rule being established based on a seri of repeated experiences. For example we may explain Indeductive argument as her umbrella, and she did not get wet.

Premises : An umbrella prevents you from getting wet in the rain. Hema too

Conclusion : In this case, we could use inductive reasoning to offer an opinio that it was probably raining.

Explanation : Our conclusion, however ecause Hema would have remained dry umbrella, or it didn't rain at all.

A deductive argument may be valid or in ay be strong or week.

Logic can include the act of reasoning nd opinions, as well as classifications as or making a conclusion. In logic the follo a conclusion :

Step 1 : The foundation of a logical ar

Step 2 : The proposition is either acc

Step 3 : Premises are the proposition

Step 4 : The argument is then built

Step 5 : Then an inference is made

Step 6 : Finally, a conclusion is dra

Generally, there are four types of l *ymbolic logic* and *Mathematical logi*

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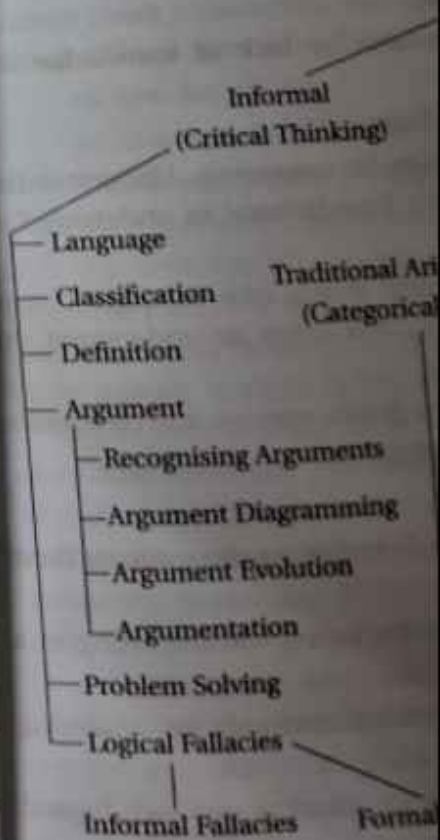


Explanation : Our conclusion, however, would not necessarily be accurate because Hema would have remained dry whether it rained as she had an umbrella, or it didn't rain at all.

A deductive argument may be valid or invalid whereas an inductive argument may be strong or week.

Logic can include the act of reasoning by humans in order to form thoughts and opinions, as well as classifications and judgments. Thus, logic is a process for making a conclusion. In logic the following steps may be involved for arriving at a conclusion :

Step 1 : The foundation of a logical argument is its proposition, or statement.

Step 2 : The proposition is either accurate (true) or not accurate (false).

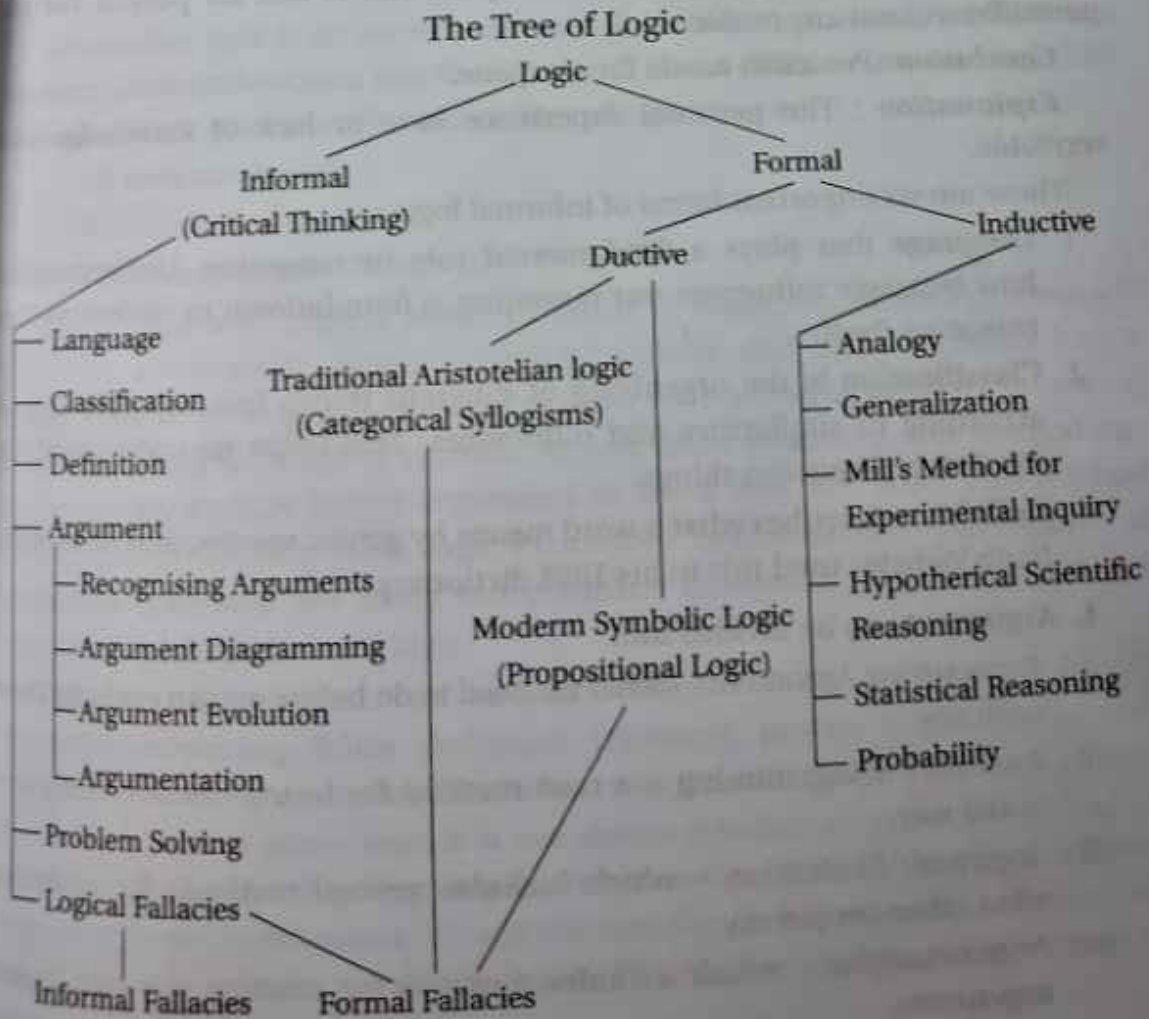
Step 3 : Premises are the propositions used to build the argument.

Step 4 : The argument is then built on premises.

Step 5 : Then an inference is made from the premises.

Step 6 : Finally, a conclusion is drawn.

Generally, there are four types of logic, such as, *Informal logic*, *Formal logic*, *Symbolic logic* and *Mathematical logic*.



1.2.1 Informal logic

Informal logic is the type of logic typically used in daily reasoning. This ty reasoning is used in our personal exchanges with others. Informal logic is u regarded as an alternative to formal or mathematical logic. It is also kno *non-formal logic.*

In North America, the rise of informal logic as it is now understood is th educational trends rooted in the 1960s — a time of social upheaval and pr (most notably, against the War in Vietnam) which produced calls for an educ relevant to the issues of the day. This fueled an interest in the logic of ever argument and the teaching of introductory logic, which was at the time ta with textbooks like the Phylosopher Irving M. Copi's popular book *Introducti Logic* (1953). The idea that informal logic should be developed as a distinct of inquiry was advocated by the two Canadian Philosophers Ralph Henry John and J. Anthony Blair at the University of Windsor. According to them info logic is " a branch of logic whose task is to develop non-formal standards, crite procedures for the analysis, interpretation, evaluation, criticism and construct of argumentation". To support this let us consider the following example :

Premises : There is no evidence that penicillin is bad for people. We u penicillin without any problem.

Conclusion : Penicillin is safe for everyone.

Explanation : The personal experience here or lack of knowledge is verifiable.

These are six important forms of informal logic :

1. **Language** that plays a fundamental role in reasoning. Understandi how language influences our reasoning is foundational to understandi reasoning itself.
2. **Classification** is the organizing of different things into separate class according to similarities and differences. This helps us understand th relationship between things.
3. **Definition** describes what a word means by genus, species, and differenti Noah Webster used this in his 1828 dictionary.
4. **Argument** can be divided into :
   (i) Recognizing Arguments – what we need to do before we can evaluate the critically
   (ii) Argument Diagramming – a neat method for laying out reasoning in visual way
   (iii) Argument Evaluation – which includes general methods for analyzing what other people say
   (iv) Argumentation – which includes methods for creating our own logica arguments.

5. **Problem Solving** is t
6. **Logical Fallacies** ar logicians have tried recognize these logi

1.2.2 Formal lo

In Formal logic we u reach a formal conclus between statements a Formal logic deals wi truth, we wish to find different types of reas

Deductive Reasoni reasoning, we can b statements that we k if they are valid. Th *Aristotelian logic* (al *logic.*

Aristotelian logic ancient Greek philos branch of logic gave

All men are m

Socrates is a

Therefore, So

Modern Symboli after the seventee vision for a more into a universal la more quickly anal of deductive reas deductive reasoni branch of modern

Inductive Rea deductive reason inductive reasoni is *probably* true certain, logicians arrives at useful around us in our developing this f

5. **Problem Solving** is the method for solving complex puzzles.
6. **Logical Fallacies** are common errors that we make in reasoning. Practical logicians have tried for thousands of years to teach ordinary people how to recognize these logical fallacies, but apparently to little avail.

1.2.2 Formal logic

In Formal logic we use deductive reasoning and the premises must be true. To reach a formal conclusion we follow the premises. It focuses on the relationship between statements as opposed to the content of any particular statement. Formal logic deals with the manipulation of truth values that is if given one truth, we wish to find another truth. Formal logic splits into two fundamentally different types of reasoning : *deductive* and *inductive*.

Deductive Reasoning is the logic of proving things for certain. With deductive reasoning, we can begin with a few true statements, and then deduce more statements that we know are also true. We can also analyze arguments to see if they are valid. The two major types of deductive reasoning are traditional *Aristotelian logic* (also called *Classical Deductive Logic*) and *modern symbolic logic*.

Aristotelian logic is an ancient method of deductive reasoning. Aristotle, the ancient Greek philosopher, was the first to talk about this kind of reasoning. This branch of logic gave us such famous lines of reasoning as :

All men are mortal.

Socrates is a man.

Therefore, Socrates is mortal.

Modern Symbolic Logic is another method of deductive reasoning developed after the seventeenth century. Gottfried Leibniz and other logicians had a vision for a more simple and useful method for translating ordinary reasoning into a universal language of symbols. Among other reasoning, this method can more quickly analyze longer arguments to see if they are valid. This method of deductive reasoning made logic even more "mathematical." This branch of deductive reasoning has many sub-branches. Computer programming is one branch of modern symbolic logic.

Inductive Reasoning is an entirely different form of reasoning from deductive reasoning. While deductive reasoning proves things for certain, inductive reasoning evaluates the evidence to determine whether an argument is *probably* true or false. Since it is not always possible to prove something for certain, logicians developed principles to reason from the evidence in a way that arrives at useful conclusions. We see the benefits of this scientific reasoning all around us in our modern technological world. Sir Frances Bacon was a leader in developing this form of reasoning.

To support this kind of reasoning, let us consider the following example :

Premises : Bicycles have two wheels. Abhimanyu is riding a bicycle.

Conclusion : Abhimanyu is riding on two wheels.

Explanation : The premises are true and so is the conclusion.

These are six important forms of inductive reasoning :

1. **Analogy** that builds arguments based on similarities between two thing
2. **Generalization** which reasons from one fact to all similar facts. This is basis for all scientific reasoning.
3. **Mill's Methods for Experimental Inquiry** were developed by John St Mill to investigate the causes of scientific phenomena.
4. **Hypothetical Scientific Reasoning** is used when scientists form theo about the world and test their theories with experiments.
5. **Statistical Reasoning** creates and interprets statistics about our world draws conclusions from those statistics.
6. **Probability** is a science closely related to statistical reasoning. It tries predict the future based on our knowledge of past experiences.

1.2.3 Symbolic logic

This kind of logic is the simplest of all kinds of logic that deals with the manne how symbols relate to each other. It assigns symbols to verbal reasoning in orde to be able to check the truthfullness of the statements through a mathematic process. We typically see this type of logic used in calculus and theoretic computer science. The idea of this logic was first developed by Jeorge Boole nineteenth century.

To support this kind of logic, let us consider the following example :

Premises : If all mammals feed their babies milk from the mother (p). If a cows feed their babies mother's milk (q). All cows are mammals (r).

Here, the symbol "$\wedge$" means "and" whereas the symbol "$\Rightarrow$" means "implies."

Conclusion : $p \wedge q \Rightarrow r$

Explanation : Proposition p and proposition q lead to the conclusion, r. If al mammals feed their babies milk from the mother and all cows feed their babies mother's milk, it implies all cows are mammals.

1.2.4 Mathematical logic

In mathematical logic, we apply formal logic to mathematics. This type of logic is a part of the basis for the logic used in computer sciences. Mathematical logic and symbolic logic are often used interchangeably. The founders of Mathematical logic such as Leibniz, Boole and Frege, hoped to use it for common sense facts and reasoning.

Aristotle identified some simple patterns in human reasoning, and Leibniz dreamt of reducing reasoning to calculation. As a viable mathematical subject,

however, logic is relatively recent : the 19th Cantor, Dedekind, Frege, Peano, C.S. Peirce, we see their work as leading to boolean a predicate logic, as clarifying the founda systems, and as introducing suggestive sy

In this book we shall deal with Math fields :

1. Model theory.
2. Proof theory.
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Now, we shall discuss about tru depicts how the truth or falsity of a on the truth or falsity of the simp constructed. Then we shall discuss simple statements (propositions)

1.3 Truth value of a

A proposition has two possibl a proposition is true, then the tr T or 1. If the proposition is false or 0.

Example 1.2 : Let us consi

1. p : 3 + 8 =12. This propos
2. q : "distance travelled by proposition is true. So, t

The symbols p, q, r, ... etc. take two different values T o called *propositional logic* or

The propositions which (to be introduced in the propositions. The mathema more simple propositions w or complex) propositions.

however, logic is relatively recent : the 19th century pioneers were Bolzano, Boole, Cantor, Dedekind, Frege, Peano, C.S. Peirce, and E. Schröder. From our perspective we see their work as leading to boolean algebra, set theory, propositional logic, predicate logic, as clarifying the foundations of the natural and real number systems, and as introducing suggestive symbolic notation for logical operations.

In this book we shall deal with Mathematical logic that is divided into four fields :

1. Model theory.
2. Proof theory.
3. Recursion theory, also known as computability theory.
4. Set theory.

We shall discuss every field of Mathematical logic in details in different chapters of this book.

Propositional logic or calculus is a branch of mathematical logic which studies the logical relationships between propositions (or statements) taken as a whole, and connected via logical connectives (logical operators). We shall discuss the concepts of Propositional logic in details in Chapter 2.

Now, we shall discuss about truth tables which is a mathematical table that depicts how the truth or falsity of a compound statement (propoditions) depends on the truth or falsity of the simple statements (propositions) from which it is constructed. Then we shall discuss about the logical connectives that connect the simple statements (propositions) to form compound statements (propositions).

1.3 Truth value of a proposition

A proposition has two possible values True (T) or False (F). This means that if a proposition is true, then the truth value of that proposition is true, denoted by T or 1. If the proposition is false, the truth value is said to be false, denoted by F or 0.

Example 1.2 : Let us consider the following propositions :

1. p : 3 + 8 =12. This proposition is false. So, p takes the truth value F or 0.
2. q : "distance travelled by an object is the product of its speed and time". This proposition is true. So, the truth value of q is T or 1.

The symbols $p, q, r, \ldots$ etc. are also known as propositional variables since they take two different values T or F. The field of logic that deals with propositions is called *propositional logic* or *propositional calculus*.

The propositions which do not contain any logical operators or connectives (to be introduced in the section 1.5) are called simple (primary or atomic) propositions. The mathematical statements constructed by connecting one or more simple propositions with logical operators are called compound (molecular or complex) propositions.

1.4 Truth Table

Truth table is an array on which all possible truth values of compo statements are displayed, through the display of all possible combinations of truth values of their simple components. It is used to test the validity of m deductive arguments.

Thus, the table displays all the truth vales assumed by proposition example for the proposition p the truth table is

Table 1.1

p
T
F

In particular, truth tables can be used to show whether a propositio expression is true for all legitimate input values, that is, logically valid.

The concept of Truth tables and the method of their constructions was invent by the German philosopher Ludwig Josef Johann Wittgenstein along with t American philosopher Emil Leon Post. They invented truth tables to accompa the calculus into which modern analytic philosophers had transformed logi They ensured a way to quickly solve for the truth of complex logical propositio in the modern logical systems.

1.5 Logical connectives

Simple propositions may be modified or combined in various ways to for new propositions. Different connectives or operators are used to connect thes propositions. The five common Logical Connectives or Operators are :

1. Logical Conjunction (AND)
2. Logical Disjunction
   (i) Inclusive OR (ii) Exclusive OR
3. Logical Negation (NOT)
4. Logical Implication (Conditional)
5. Logical Biconditional (Double Implication)

These connectives are discussed below :

1.5.1 Conjunction (logical AND)

Let p and q are two propositions, then the proposition "p and q" is called conjunction of p and q. It is denoted by $p \wedge q$ (read as p and q). This is a compound proposition which is true when both p and q are true (T) and is false (F) otherwise. It is notable that conjunction is a binary connective that has the same properties as intersection in Set theory. The word "but" may be used instead of "and" for a conjunction in a compound statement.

Example 1.3. Let p : "It is ra
the conjunction of p and q
The truth table of the conjun

Example 1.4. Consider the
1. Mumbai is in India and
2. Taj Mahal is red and 2 +
3. Kolkata is the capital of
4. Bangalore is in Gujrat a

The statements (1) and (3)

1.5.2 Disjunction (lo

In natural language, there *Exclusive* as discussed below

(i) **Inclusive OR :** Let p and is called disjunction of a compound propositio is true (T) otherwise. D property as union in Se

Example 1.5. Let p : propositions. Then the intelligent".

The truth table of the di

Example 1.6. Conside
1. Mumbai is in India
2. Taj Mahal is red or
3. Kolkata is the capit
4. Bangalore is in Guj

The statements (1), (2)

Example 1.3. Let p : "It is raining", q : "It is cold" be two propositions". Then the conjunction of p and q is $p \wedge q$: "It is raining and cold".
The truth table of the conjunction $p \wedge q$ is [KU, 2019]

Table 1.2

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Example 1.4. Consider the following statements :

1. Mumbai is in India and $2 + 2 = 4$
2. Taj Mahal is red and $2 + 2 = 4$
3. Kolkata is the capital of West Bengal and $2 + 2 = 4$
4. Bangalore is in Gujrat and $2 + 2 = 5$

The statements (1) and (3) are true (T) but the statements (2) and (4) are false (F)

1.5.2 Disjunction (logical OR)

In natural language, there are two different uses of "or", such as, *Inclusive* and *Exclusive* as discussed below :

(I) **Inclusive OR :** Let p and q are two propositions, then the proposition "p or q" is called disjunction of p and q. It is denoted by $p \vee q$ (read as p or q). This is a compound proposition which is false when both p and q are false (F) and is true (T) otherwise. Disjunction is a binary connective that has the same property as union in Set theory. It is also called 'inclusive or'.

Example 1.5. Let p : "Shyam is tall", q : "Shyam is intelligent" be two propositions. Then the disjunction of p and q is $p \vee q$: "Shyam is tall or he is intelligent".
The truth table of the disjunction $p \wedge q$ is [KU, 2019]

Table 1.3

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Example 1.6. Consider the following statements :

1. Mumbai is in India or $2 + 2 = 4$
2. Taj Mahal is red or $2 + 2 = 4$
3. Kolkata is the capital of West Bengal or $2 + 2 = 5$
4. Bangalore is in Gujrat or $2 + 2 = 5$

The statements (1), (2) and (3) are true (T) but the statement (4) is false (F)

(ii) **Exclusive OR :** Let p and q are two propositions, then the proposition "p or q" is called exclusive disjunction of p and q if it is true when exactly one of p and q are true (T), otherwise it is False. It is denoted by $p \oplus q$, where $\oplus$ is a binary connective.

The truth table of the disjunction $p \oplus q$ is

Table 1.4

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

1.5.3 Negation (NOT)

Given any proposition p, another proposition formed by writing "It is not the case that" or "It is false that" before p or by inserting the word 'not' suitably in p is called negation of p. It is denoted by $\sim p$ or $\neg p$ (read as not p). This is a compound proposition which is true (T) when p is true and false (F) when p is false. So, a statement in a natural language may be negated symbolically in several ways keeping in view the same meaning for every negated statement.

Example 1.7. Let p : "It is hot today" be any proposition, then $\sim p$ is any one of the following statements.

1. It is not the case that today is hot
2. It is false that today is hot
3. It is not hot today

The truth table for negation $\sim p$ is

Table 1.5

p	$\sim p$
T	F
F	T

Example 1.8. (a) If p : "$8 + 1 = 5$", then $\sim p$: "$8 + 1 \neq 5$"

(b) Let p : "$\sqrt{2}$ is a ratinal number" be any proposition, then $\sim p$ is "$\sqrt{2}$ is not a ratinal number".

Note

For a proposition p, negation of a negation is the affirmation. So, $\sim(\sim p) \equiv p$.

1.5.4 Implication (Conditional)

Let p and q are two propositions, then the proposition "if p then q" is called conditional or implication proposition of p and q. It is denoted by $p \rightarrow q$ or

$p \Rightarrow q$ (read as p implies q). The state
or **hypothesis** and the statement q is
compound proposition which is false
(T) otherwise.

Example 1.9. Let p : "Rahim
Examination" be two proposit
$p \rightarrow q$: "If Rahim works hard t
The truth table of $p \rightarrow q$ is

Example 1.10. Let p and q
p : Two triangles are congr
q : Two triangles are simil
Then $p \rightarrow q$ can be expre
1. If two triangles are co
2. Two triangles are con
3. Two triangles are con
4. The necessary condi
congruent

1.5.5 Biconditiona

Let p and q are two pro
"if p then q and if q then
denoted by $p \leftrightarrow q$ (read
which is true when p and
It can be easily verifie
and $q \rightarrow p$ are true. So, th
Alternatively, $p \leftrightarrow q$
sufficient for q".
i.e., $p \leftrightarrow q$ is $(p \rightarrow q)$

Example 1.11. Let
Then the biconditi
$p \leftrightarrow q$: "Birds fly if
is clear and if the

$p \Rightarrow q$ (read as p implies q). The statement p is called the **antecedent** or **premise** or **hypothesis** and the statement q is called **consequent** or **conclusion**. This is a compound proposition which is false when p is true (T) and q is false (F) and true (T) otherwise. [KU, 2019]

Example 1.9. Let p : "Rahim works hard", q : "Rahim comes first in Examination" be two propositions. Then the implication of p and q is

$p \rightarrow q$: "If Rahim works hard then he comes first in Examination".

The truth table of $p \rightarrow q$ is [VU, 2018]

Table 1.6

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Example 1.10. Let p and q be two propositions, such that

p : Two triangles are congruent

q : Two triangles are similar

Then $p \rightarrow q$ can be expressed as any one of the statements

1. If two triangles are congruent, then they are similar
2. Two triangles are congruent implies that they are similar
3. Two triangles are congruent only if they are similar
4. The necessary condition for two triangles to be similar is that they are congruent

1.5.5 Biconditional (Double Implication)

Let p and q are two propositions, then the proposition "p if and only if q" or "if p then q and if q then p" is called biconditional proposition of p and q. It is denoted by $p \leftrightarrow q$ (read as p if and only if q). This is a compound proposition which is true when p and q has the same truth values and is false otherwise.

It can be easily verified that $p \leftrightarrow q$ is true when both the conditionals $p \rightarrow q$ and $q \rightarrow p$ are true. So, the symbol $\leftrightarrow$ is a combination of the symbols $\rightarrow$ and $\leftarrow$

Alternatively, $p \leftrightarrow q$ is also expressed as "p iff q" and "p is necessary and sufficient for q".

i.e., $p \leftrightarrow q$ is $(p \rightarrow q) \wedge (q \rightarrow p)$

Example 1.11. Let p : "Birds fly" and q : "Sky is clear" be two propositions. Then the biconditional of p and q is

$p \leftrightarrow q$: "Birds fly if and only if the sky is clear" or "If the birds fly then the sky is clear and if the sky is clear then the birds fly".

The truth table of $p \leftrightarrow q$ is [VU, 20

Table 1.7

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Note

The operators $\wedge, \vee, \sim, \rightarrow$ and $\leftrightarrow$ are known as Connectives.

Now, associated with "$p \rightarrow q$" we can define following three condition connectives.

1.6 Converse, Inverse and Contrapositive

Given an if-then statement "if p then q," we can create three related statements

1. Converse
2. Inverse
3. Contrapositive

1.6.1 Converse

We know that a conditional statement consists of two parts, a hypothesis in the "if" clause and a conclusion in the "then" clause. For example, let us consider the propositions

p : It rains

q : The boys cancel school.

Here, the proposition p is hypothesis and the proposition q is the conclusion. The conditional statement $p \rightarrow q$ is "If it rains then the boys cancel school".

To form the converse of the conditional statement, we must interchange the hypothesis p and the conclusion q. Thus, the converse of "If it rains then the boys cancel school" is "If the boys cancel school, then it rains". The converse of $p \rightarrow q$ is symbolically written as $q \rightarrow p$. In general, the truth of the statement $p \rightarrow q$ says nothing about the truth of its converse statement $q \rightarrow p$, unless the antecedent p and the consequent q are logically equivalent. For example let us consider the following propositions :

p : Ashok is rich

q : Ashok lives in a bunglow

Here, the statement $p \rightarrow q$ means "if Ashok is rich, then he lives in a bunglow" Consequently, the converse statement $q \rightarrow p$ is "if Ashok lives in a bunglow, then he is rich". We observe that $p \rightarrow q$ is true while $q \rightarrow p$ may not necessarily be true.

The truth table of $q \rightarrow p$ is

Table 1.8

p	q
T	T
T	F
F	T
F	F

1.6.2 Inverse

To form the inverse of the conditional st of both the hypothesis p and the conclusio

The inverse of "If it rains then the boys then the boys do not cancel school". The in as $\sim p \rightarrow \sim q$.

The truth table of $\sim p \rightarrow \sim q$ is

Table

p	q
T	T
T	F
F	T
F	F

Note

The inverse and converse of a condition each other.

1.6.3 Contrapositive

To form the contrapositive of the co the hypothesis p and the conclusion q

The contrapositive of "If it rains, the cancel school, then it does not rain." T written as $\sim q \rightarrow \sim p$.

The truth table of $\sim q \rightarrow \sim p$ is

Ta

p
T
T
F
F

Contrapositive statement is the negation. Several mathematical the

The truth table of $q \to p$ is

Table 1.8

p	q	$p \to q$
T	T	T
T	F	T
F	T	F
F	F	T

1.6.2 Inverse

To form the inverse of the conditional statement, we must take the negation of both the hypothesis p and the conclusion q.

The inverse of "If it rains then the boys cancel school" is "If it does not rain, then the boys do not cancel school". The inverse of $p \to q$ is symbolically written as $\sim p \to \sim q$.

The truth table of $\sim p \to \sim q$ is

Table 1.9

p	q	$\sim p \to \sim q$
T	T	T
T	F	T
F	T	F
F	F	T

Note

The inverse and converse of a conditional statement are logically equivalent to each other.

1.6.3 Contrapositive

To form the contrapositive of the conditional statement, we must interchange the hypothesis p and the conclusion q of the inverse statement.

The contrapositive of "If it rains, then the boys cancel school" is "If they do not cancel school, then it does not rain." The contrapositive of $p \to q$ is symbolically written as $\sim q \to \sim p$.

The truth table of $\sim q \to \sim p$ is

Table 1.10

p	q	$\sim q \to \sim p$
T	T	T
T	F	F
F	T	T
F	F	T

Contrapositive statement is the combination of inversion, conversion and negation. Several mathematical theorems may be proved by using the concept

of contrapositive known as "*proof by contradiction*" which is a very po... method and it is discussed elaborately in chapter 2.

Example 1.12. Given any conditional statement $p \to q$, the three re... statements can be written as follows :

Statement : If x is positive, then $x \neq 0$

Here, p : If x is positive, q : $x \neq 0$

Converse	$q \to p$: If $x \neq 0$, then x is positive
Inverse	$\sim p \to \sim q$: If x is negative, then $x = 0$
Contrapositive	$\sim q \to \sim p$: If $x = 0$, then x is negative

Examples 1.13. Given any conditional statement $p \to q$, the three re... statements can be written as follows :

Statement : If two angles are congruent, then they have the same mea...

Here, p : If two angles are congruent, q : they have the same measure

Converse	$q \to p$: If two angles have the same measure, then they... congruent.
Inverse	$\sim p \to \sim q$: If two angles are not congruent, then they do not h... the same measure.
Contrapositive	$\sim q \to \sim p$: If two angles do not have the same measure, then ... are not congruent.

Example 1.14. Given any conditional statement $p \to q$, the three rel... statements can be written as follows :

Statement : If $6x - 3 = 9$, then $x = 2$

Here, p : $6x - 3 = 9$, q : $x = 2$

Converse	$q \to p$: If $x = 2$, then $6x - 3 = 9$.
Inverse	$\sim p \to \sim q$: $6x - 3 \neq 9$, then $x \neq 2$
Contrapositive	$\sim q \to \sim p$: If $x \neq 2$, then $6x - 3 \neq 9$.

Note

If a statement is true, then the contrapositive is also logically true. If t... converse is true, then the inverse is also logically true.

1.7 Propositional Formula or Statement Formula

A compound proposition $P = f(p, q, r, ...,)$ obtained by connecting one or m... of the variables (simple propositions) $p, q, r, ...$ by the connectives $\wedge, \vee, \sim, \to$ o... is called a propositional formula or statement formula.

Example 1.15. Let $P = f(p, q) = \sim(\sim p \wedge q)$ be a propositional formula... statement formula of the variables p and q.

The truth table of a propositional st...

Tab...

p	q	$\sim p$
T	T	F
T	F	F
F	T	T
F	F	T

Example 1.16. Let f $(p, q, r) = p \wedge$ ... statement formula of the variabl...

The truth table of the proposition...

Ta...

p	q	r	$\sim q$	$p \wedge$ (
T	T	T	F	
T	T	F	F	
T	F	T	T	
T	F	F	T	
F	T	T	F	
F	T	F	F	
F	F	T	T	
F	F	F	T	

Note

The number of rows in a truth ta... n is the number of variables in t... the above truth table of the prop... number rows is $2^2 = 4$, since, th... the truth table of the propositio... rows, since, the formula contain...

1.7.1 Duality law

The dual of a compound prop... and T, F is the proposition obtai... T by F and each F by T, where T a... tautologies and contradictions re...

1.8 Precedence Rule

While calculating the truth va... than one logical operator, the r... odering of logical operators nam... to the formula of propositions th...

of contrapositive known as "*proof by contradiction*" which is a very po... method and it is discussed elaborately in chapter 2.

Example 1.12. Given any conditional statement $p \to q$, the three re... statements can be written as follows :

Statement : If x is positive, then $x \neq 0$

Here, p : If x is positive, $q : x \neq 0$

Converse	$q \to p$: If $x \neq 0$, then x is positive
Inverse	$\sim p \to \sim q$: If x is negative, then $x = 0$
Contrapositive	$\sim q \to \sim p$: If $x = 0$, then x is negative

Examples 1.13. Given any conditional statement $p \to q$, the three re... statements can be written as follows :

Statement : If two angles are congruent, then they have the same meas...

Here, p : If two angles are congruent, q : they have the same measure

Converse	$q \to p$: If two angles have the same measure, then they ... congruent.
Inverse	$\sim p \to \sim q$: If two angles are not congruent, then they do not h... the same measure.
Contrapositive	$\sim q \to \sim p$: If two angles do not have the same measure, then t... are not congruent.

Example 1.14. Given any conditional statement $p \to q$, the three rel... statements can be written as follows :

Statement : If $6x - 3 = 9$, then $x = 2$

Here, $p : 6x - 3 = 9$, $q : x = 2$

Converse	$q \to p$: If $x = 2$, then $6x - 3 = 9$.
Inverse	$\sim p \to \sim q$: $6x - 3 \neq 9$, then $x \neq 2$
Contrapositive	$\sim q \to \sim p$: If $x \neq 2$, then $6x - 3 \neq 9$.

Note

If a statement is true, then the contrapositive is also logically true. If t... converse is true, then the inverse is also logically true.

1.7 Propositional Formula or Statement Formula

A compound proposition $P = f(p, q, r, ...)$ obtained by connecting one or m... of the variables (simple propositions) $p, q, r, ...$ by the connectives $\wedge, \vee, \sim, \to$ or... is called a propositional formula or statement formula.

Example 1.15. Let $P = f(p, q) = \sim(\sim p \wedge q)$ be a propositional formula... statement formula of the variables p and q.

The truth table of a propositional st...

Tab...

p	q	$\sim p$
T	T	F
T	F	F
F	T	T
F	F	T

Example 1.16. Let $f(p, q, r) = p \wedge$... statement formula of the variabl...

The truth table of the proposition...

Ta...

p	q	r	$\sim q$	$p \wedge$ (
T	T	T	F	
T	T	F	F	
T	F	T	T	
T	F	F	T	
F	T	T	F	
F	T	F	F	
F	F	T	T	
F	F	F	T	

Note

The number of rows in a truth ta... n is the number of variables in t... the above truth table of the prop... number rows is $2^2 = 4$, since, th... the truth table of the propositio... rows, since, the formula contain...

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The dual of a compound prop... and T, F is the proposition obtai... T by F and each F by T, where T a... tautologies and contradictions re...

1.8 Precedence Rule

While calculating the truth va... than one logical operator, the r... odering of logical operators nam... to the formula of propositions th...

The truth table of a propositional statement $f(p, q) = \neg(\neg p \wedge q)$ is

Table 1.11

p	q	$\neg p$	$\neg p \wedge q$	$\neg(\neg p \wedge q)$
T	T	F	F	T
T	F	F	F	T
F	T	T	T	F
F	F	T	F	T

Example 1.16. Let $f(p, q, r) = p \wedge (\neg q) \vee r \rightarrow p$ is a propositional formula or statement formula of the variables p, q, r.

The truth table of the propositional statement $f(p, q, r) = p \wedge (\neg q) \vee r \rightarrow p$ is

Table 1.12

p	q	r	$\neg q$	$p \wedge (\neg q)$	$p \wedge (\neg q) \vee r$	$p \wedge (\neg q) \vee r \rightarrow p$
T	T	T	F	F	T	T
T	T	F	F	F	F	T
T	F	T	T	T	T	T
T	F	F	T	T	T	T
F	T	T	F	F	T	F
F	T	F	F	F	F	T
F	F	T	T	F	T	F
F	F	F	T	F	F	T

Note

The number of rows in a truth table of a propositional statement is 2^n where n is the number of variables in the propositional statement. For example in the above truth table of the propositional formula $P = f(p, q) = \neg(\neg p \wedge q)$, the number rows is $2^2 = 4$, since, the formula contains two variables p and q. In the truth table of the propositional statement $p \wedge (\neg q) \vee r \rightarrow p$ there are $2^3 = 8$ rows, since, the formula contains three variables p, q and r.

1.7.1 Duality law

The dual of a compound proposition that contains the logical operators $\vee$, $\wedge$ and T, F is the proposition obtained by replacing each $\vee$ by $\wedge$, each $\wedge$ by $\vee$, each T by F and each F by T, where T and F represent compound propositions that are tautologies and contradictions respectively. Then dual of $\neg p \wedge q$ is $\neg p \vee q$.

1.8 Precedence Rule of Logical Operators

While calculating the truth value of compound propositions involving more than one logical operator, the rule of operator precedence is used. This is the ordering of logical operators named as 'the rule of precedence'. This rule is applied to the formula of propositions that has no paranthesis. The negation '¬' operator

has the highest precedence among all other logical operators. The ord... precedence is shown in the table below :

Table 1.13

Operator	Precedence
~	1
∧	2
∨	3
→	4
↔	5

So, the order of precedence is : the operator negation '~' is performed at t... conjuction '∧' is performed next, and then disjunction '∨' is performed. The... the end comes the order of precedence of implication '→' and biconditional'... All operators are right associative.

Let us consider the following example to clear this rule.

Example 1.17. Let us find the truth value of $p \to q \wedge \sim r \leftrightarrow r \vee q$.

To find the truth value of the compound proposition, we need to use the r... of precedence. We write the statement with paranthesis as $(p \to (q \wedge (\sim r)) \leftrightarrow (r \vee$...

The truth table for $(p \to (q \wedge (\sim r)) \leftrightarrow (r \vee q)$ is

Table 1.14

p	q	r	$\sim r$	$q \wedge (\sim r)$	$r \vee q$	$p \to (q \wedge (\sim r))$	$p \to (q \wedge (\sim r)) \leftrightarrow (r \vee$ ...
T	T	T	F	F	T	F	F
T	T	F	T	T	T	T	T
T	F	T	F	F	T	F	F
T	F	F	T	F	F	F	T
F	T	T	F	F	T	T	T
F	T	F	T	T	T	T	T
F	F	T	F	F	T	T	T
F	F	F	T	F	F	T	F

1.9 Introduction to Formal Language

In order to communicate in any language such as Bengali, Hindi, English, French, etc. which are called natural languages, we must know the letters of alphabet and then know how the letters should be combined in a meaningful sentence. The set of rules of syntax that lead to automatic translation of a natural language to another is referred as *formal language.*

A simple method of developing a formal language is *grammar*, which is an algebraic structure that describes the process by which instances of a language can be constructed. It gives mathematical structural descriptions of sentences for accepting or rejecting strings (words) in a language. This determines the membership criterion of a string in a language.

1.9.1 Language : The set theory of strings

A language is created using elements such as Alphabet and s...

Alphabet : An Alphabet denoted mathematically by 'Σ' is a set of *symbols*. The word symbol used here is an abstract expressed by using English letters or non-negative integers.

Example 1.18 : The following are examples of Alphabets

(i) Binary Alphabet : $\Sigma = \{0, 1\}$,

(ii) Decimal Alphabet : $\Sigma = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

(iii) An alphabet of lower case letters : $\Sigma = \{a, b, ..., z\}$

Strings : A string or word denoted mathematically by 'w' ... symbols taken from some alphabet Σ.

Example 1.19 : The following are examples of strings

(i) For the alphabet $\Sigma = \{x, y\}$, x, $xyxy$, $xxxyy$, $xyxyxyx$... etc. are strings

(ii) For the alphabet $\Sigma = \{0,1\}$, 0010110 is a string

Length of String : The length of string is the number ...

Example 1.20 : The following are examples of length...

(i) If for alphabet $\Sigma = \{a, b\}$, the string $w = ababaa$... string is $|w| = 7$

(ii) If for alphabet $\Sigma = \{0, 1\}$, the string $|w| = 1101$, th... is $|w| = 4$

Empty string : Empty string or null string is the st... of symbols. The null string can be denoted with Lam...

Kleene Star : The set of all strings of all possib... string on the alphabet is called the Kleene star deno...

Example 1.21 : If $\Sigma = \{a, b\}$ then

$\Sigma^* = \{a, b\}^* = \{\lambda\lambda, a, b, aa, ab, ba, aaa, abb, aab$...

Kleene Closure / Plus : The set Σ^* is the infinite ... possible lengths over Σ excluding λ.

Example 1.22 : If $\Sigma = \{a, b\}$, then

$\Sigma^* = (a, b)^* = \{a, b, aa, ab, ba, aaa, abb, aab$, ...

So, the difference between sets Σ^* and Σ^+ is ... because $\lambda \in \Sigma^*$ only when $n = 0$. Thus, $\Sigma^* = \Sigma^+$ ...

Note

The symbol λ is not an element of any alp... considered as blank (space) found in any alph...

(i) $\{\lambda\} \not\subset \Sigma$, $\lambda \notin \Sigma$ (ii) $|\{\lambda\}| = 1 \neq 0 = |\phi|$

(iii) We have $\phi \neq \{\lambda\}$, as the language ϕ doe... contains a string, namely λ. Also it is evid...

1.9.1 Language : The set theory of strings

A language is created using elements such as Alphabet and strings.

Alphabet : An Alphabet denoted mathematically by 'Σ' is a non-empty finite set of *symbols*. The word symbol used here is an abstract entity frequently expressed by using English letters or non-negative integers.

Example 1.18 : The following are examples of Alphabets

(i) Binary Alphabet : $\Sigma = \{0, 1\}$,

(ii) Decimal Alphabet : $\Sigma = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

(iii) An alphabet of lower case letters : $\Sigma = \{a, b, ..., z\}$

Strings : A string or word denoted mathematically by 'w' is finite sequence of symbols taken from some alphabet Σ.

Example 1.19 : The following are examples of strings

(i) For the alphabet $\Sigma = \{x, y\}$, x, $xyxy$, $xxxyy$, $xyxyxyxyx$, $xyxyxyxyxyxyxyxyxy$ etc. are strings

(ii) For the alphabet $\Sigma = \{0,1\}$, 0010110 is a string

Length of String : The length of string is the number of letters in the string.

Example 1.20 : The following are examples of length of strings

(i) If for alphabet $\Sigma = \{a, b\}$, the string $w = ababaab$, then the length of the string is $|w| = 7$

(ii) If for alphabet $\Sigma = \{0, 1\}$, the string $|w| = 1101$, then the length of the string is $|w| = 4$

Empty string : Empty string or null string is the strings with zero occurrence of symbols. The null string can be denoted with Lambda 'λ' and $|\lambda| = 0$.

Kleene Star : The set of all strings of all possible lengths including empty string on the alphabet is called the Kleene star denoted by Σ^*.

Example 1.21 : If $\Sigma = \{a, b\}$ then

$\Sigma^* = \{a, b\}^* = \{\lambda\lambda, a, b, aa, ab, ba, aaa, abb, aab, aba, ...\}$

Kleene Closure / Plus : The set Σ^* is the infinite set of all possible strings of all possible lengths over Σ excluding λ.

Example 1.22 : If $\Sigma = \{a, b\}$, then

$\Sigma^+ = \{a, b\}^+ = \{a, b, aa, ab, ba, aaa, abb, aab, aba, ...\}$

So, the difference between sets Σ^* and Σ^+ is the presence of the element λ because $\lambda \in \Sigma^*$ only when $n = 0$. Thus, $\Sigma^+ = \Sigma^* - \{\lambda\}$

Note

The symbol λ is not an element of any alphabet Σ and it should not be considered as blank (space) found in any alphabet. Thus

(i) $\{\lambda\} \subset \Sigma$, $\lambda \notin \Sigma$ (ii) $|\{\lambda\}| = 1 \neq 0 = |\phi|$

(iii) We have $\phi \neq \{\lambda\}$, as the language ϕ does not contain any string but $\{\lambda\}$ contains a string, namely λ. Also it is evident that $\phi = 0$ but $\{\lambda\} = 1$.

1.9.2 Operations on Strings

The basic operation for strings is the *concatenation* operation. concatenation of strings is a string formed by writing the letters (symbols) string first followed by letters of second string. For example if w_1= dog and house are two strings of length 3 and 5 respectively, then the concatenation strings w_1 and w_2 written as w_1w_2 = doghouse is a string of length $3 + 5 = 8$

Example 1.23 : If $w_1 = abb$ and $w_2 = ba$, then $w_1w_2 = abbba$ and $w_2w_1 =$

Naturally, concatenating with the empty string results in no change string. Formally, for any string w, we have that $w\lambda = w$. As such $\lambda\lambda = \lambda$.

For a string w, the string w^* is a substring of w if the string w^* ap continuously in w.

Example 1.24 : For $w = abcdef$ we have that bcd is a substring of w, but not a substring of w.

A string w^* is a suffix of w if it is a substring of w appearing in the end Similarly, w^{**} is a prefix of w if is a substring of w appearing in the beginning

Example 1.25 : For $w = abcdef$ we have that abc is a prefix of w, and de suffix of w.

1.9.3 Language

We have now got acquainted with the general notion of strings that are b elements of a language. The notion of a language in a broad spectrum is collection of strings over an alphabet. Thus we define a *language* over an alph Σ as a subset of Σ^*. It is denoted by L. This includes the subset ϕ called the em language. A string in a language L is called a sentence of Σ.

Since languages are sets, we can apply various well known set operati such as union, intersection, complement, difference on languages. The notio concatenation of strings can be extended to languages as follows.

The concatenation of a pair of languages L_1, L_2 is

$$L_1L_2 = \{a, b : a \in L_1 \wedge b \in L_2\}$$

Example 1.26 : For $L_1 = \{b, ba, bab\}$ and $L_2 = \{\lambda, b, bb, abb\}$ we have

$$L_1L_2 = \{b, ba, bb, bab, bbb, babb, baabb, babbb, bababb\}$$

Notes

(i) Since concatenation of strings is associative, so is the concatenation languages. That is, for all languages L_1,L_2 and L_3,

$$(L_1L_2)L_3 = L_1(L_2L_3)$$

(ii) The number of strings in L_1L_2 is always less than or equal to the product individual numbers, i.e. $|L_1L_2| \leq |L_1||L_2|$

(iii) $L_1 \subseteq L_1L_2$ if and only if $\lambda \in L_2$

1.9.4 Object and Meta Language

The object Languages are the Formal Languag Logic, Predicate Logic etc. We need to use a La anything about an object language.

In logic and linguistics, a metalanguage u language, called the object language. The differ often make use of italics, quotation marks, or not done in an Object Language. Metalangua logic and linguistics to analyze or describe t language itself.

Illustrative

1. Let p : 'Today is Sunday' and q : 'It sentences which describe the com (i) $q \wedge \neg p$ (ii) $\neg(p \vee q)$ (iii) $p \vee q$

Solution (i) It is hot and today it is not S
(ii) It is not true that Today is Sunday or
(iii) Today is Sunday or it is hot

2. Find a propositional formula tha "If I win the lottery or pass the

Solution Let us consider the propositi
q : I pass the examination
r : I am happy
The statement in logical form is $(p$

3. Let p : 'He is intelligent' and each of the following state propositions p and q.
(i) He is fat but not intellig
(ii) He is neither fat nor int
(iii) He is intelligent or he i
(iv) It is not true that he is
(v) It is not true that he is
(vi) If he is intelligent ther

Solution (i) $q \wedge \neg p$

1.9.4 Object and Meta Language

The object Languages are the Formal Language, Propositional Logic, Sentence Logic, Predicate Logic etc. We need to use a Language, if we are going to state anything about an object language.

In logic and linguistics, a metalanguage is a language used to describe a language, called the object language. The difference is that in a metalanguage we often make use of italics, quotation marks, or writing on a separate line which is not done in an Object Language. Metalanguage is an artificial language used in logic and linguistics to analyze or describe the sentences or elements of object language itself.

Illustrative Examples

1. **Let p : 'Today is Sunday' and q : 'It is hot' be two propositions. Write the sentences which describe the compound propositions**
(i) $q \wedge \sim p$ (ii) $\sim(p \vee q)$ (iii) $p \vee q$

Solution (i) It is hot and today it is not Sunday
(ii) It is not true that Today is Sunday or it is hot
(iii) Today is Sunday or it is hot

2. **Find a propositional formula that represents the following propositions :**
"If I win the lottery or pass the examination, then I am happy" [CU, 2021]

Solution Let us consider the proposition p : I win the lottery
q : I pass the examination
r : I am happy
The statement in logical form is $(p \vee q) \rightarrow r$

3. **Let p : 'He is intelligent' and q : 'He is fat' be two propositions. Write each of the following statements in logical form using the simple propositions p and q.**
(i) He is fat but not intelligent
(ii) He is neither fat nor intelligent
(iii) He is intelligent or he is fat
(iv) It is not true that he is intelligent or he is fat
(v) It is not true that he is neither fat nor intelligent
(vi) If he is intelligent then he is fat

Solution (i) $q \wedge \sim p$

(ii) $\sim q \vee \sim p$

(iii) $p \vee q$

(iv) $\sim(p \vee q)$

(v) $\sim(\sim q \vee \sim p)$

(vi) $p \rightarrow q$

4. **Let p : 'I get a Bachelors degree in Science', q : 'I get a job' and r : to USA' be three propositions. Write each of the following statem in logical form using the simple propositions p, q and r. Write statements corresponding to the following compound propositio**

(i) $\sim q \rightarrow (r \wedge p)$

(ii) $(p \vee q) \leftrightarrow r$

(iii) $(p \wedge \sim q) \rightarrow \sim r$

Solution (i) If I do not get a job then I go to USA and get a bachelors degre Science

(ii) I get a Bachelors degree in Science or I get a job if and only if I go to USA

(iii) If I get a Bachelors degree in Science and I do not get a job then I do no to USA

5. **Let P and Q be the propositions "Swimming at the New Jersey shor allowed" and "Sharks have been spotted near the shore", respectiv Express each of these compound propositions as English sentence.**

(a) $\sim P \vee Q$ **(b)** $P \leftrightarrow \sim Q$

Solution (a) $\sim P \vee Q$: Swimming at the New Jersey shore is not allowed or Sha have been spotted near the shore (Swimming at the New Jersey shore is not allow as Sharks have been spotted near the shore)

(b) $P \leftrightarrow \sim Q$: Swimming at the New Jersey shore is allowed if and only if Sha have not been spotted near the shore

6. **Write each of the following statements in logical form using the simp propositions p, q and r.**

(i) A necessary condition for 't' to be prime is that either t is odd $t = 2$

(ii) A sufficient condition for a function 'f' to be continuous is that f differentiable

Solution (i) Let p : t is prime

q : t is odd

r : $t = 2$

The statement is $p \rightarrow (q \vee r)$

(ii) Let p : f is continuous

q : f is differentiable

The statement is $q \rightarrow p$

7. **What is symbolic logic? Write the following st logic : If the barometer falls, then either it wi**

Solution The use of symbols to denote propositio order to assist reasoning is called Symbolic logic. It expressions by using symbols and variables in place English or any other language, in order to remove va

Let p : barometer falls

q : it will rain

r : it will snow

The statement using symbolic logic is $p \rightarrow (q \vee$

8. **Write down the negation of the statement**

(i) Today is Wednesday

(ii) No one wants to buy my bag

(iii) Every even integer greater than 2 is

(iv) Some people have no ideas

(v) Some people have no scooter

(vi) He swims if and only if the water is

(vii) The computer program is correct correct answer for all possible sets

(viii) Kolkata is a city.

(ix) Every odd integers is divisible by

Solution (i) Today is not Wednesday

(ii) Someone wants to buy my bag

(iii) Some even integer greater than 2 is not pr

(iv) All people have ideas

(v) All people have scooter

(vi) He swims and the water is not warm or swim

(vii) The computer program is correct and it for all possible sets of input data or it possible sets of input data and the comp

(viii) Kolkata is not a city.

(ix) Some odd integer is divisible by 3

(ii) Let p : f is continuous
q : f is differentiable
The statement is $q \to p$

7. **What is symbolic logic? Write the following statements using symbolic logic : If the barometer falls, then either it will rain or it will snow.** [KU, 2019]

Solution The use of symbols to denote propositions, terms, and relations in order to assist reasoning is called Symbolic logic. It is a way to represent logical expressions by using symbols and variables in place of natural language, such as English or any other language, in order to remove vagueness in concept.

Let p : barometer falls
q : it will rain
r : it will snow
The statement using symbolic logic is $p \to (q \vee r)$

8. **Write down the negation of the statements for each of the following**

(i) Today is Wednesday
(ii) No one wants to buy my bag
(iii) Every even integer greater than 2 is not prime
(iv) Some people have no ideas
(v) Some people have no scooter [MAKAUT, 2006]
(vi) He swims if and only if the water is warm [VU, 2019]
(vii) The computer program is correct if and only if, it produces the correct answer for all possible sets of input data [VU, 2019]
(viii) Kolkata is a city. [VU, 2020]
(ix) Every odd integers is divisible by 3 [VU, 2020]

Solution (i) Today is not Wednesday
(ii) Someone wants to buy my bag
(iii) Some even integer greater than 2 is not prime.
(iv) All people have ideas
(v) All people have scooter
(vi) He swims and the water is not warm or the water is warm and he does not swim
(vii) The computer program is correct and it does not produce the correct answer for all possible sets of input data or it produces the correct answer for all possible sets of input data and the computer program is not correct
(viii) Kolkata is not a city.
(ix) Some odd integer is divisible by 3

Let p, q be statements for which the implication $p \to q$ is false. Deter... the truth values for (i) $p \wedge q$ (ii) $\sim q \to \sim p$

Solution The truth table for $p \to q$ is

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

It is clear that when $p \to q$ is F, the truth vales of p and q are respectively T an...
So, the truth value of (i) $p \wedge q$ is F (ii) $\sim q \to \sim p$ is F

If p is true and q is False, then find the truth value of $(p \wedge q) \to (p \vee$ ... [MAKAUT, 20...

Solution If p assumes the truth value T and q assumes the truth value F, th...
m the truth table of conjunction the truth value of $p \wedge q$ is F and from the tru...
le of disjunction the truth value of $p \vee q$ is T.
So, the truth value of $(p \wedge q) \to (p \vee q)$ is T

Construct the truth table for each of the following compou... propositions :

(i) $(p \vee q) \to (p \wedge q)$
(ii) $(p \to q) \to (q \to p)$ [VU, 201...
(iii) $(q \to \sim p) \leftrightarrow (p \leftrightarrow q)$
(iv) $(p \to q) \leftrightarrow (\sim q \to \sim p)$
(v) $\sim (p \vee \sim q) \to \sim p$ [BU, 201...
(vi) $((\sim p \to q) \wedge (r \to p)) \leftrightarrow (\sim q \vee r)$ [CU, 202...

Solution In the compound proposition $(p \vee q) \to (p \wedge q)$ two simple propositio... q are involved. So, there must be $2^2 = 4$ rows in the truth table.

Truth table for $(p \vee q) \to (p \wedge q)$ is

p	q	$p \vee q$	$p \wedge q$	$(p \vee q) \to (p \wedge q)$
T	T	T	T	T
T	F	T	F	F
F	T	T	F	F
F	F	F	F	T

Truth table for $(p \to q) \to (q \to p)$ is

p	q	$p \to q$	$q \to p$	$(p \to q) \to (q \to p)$
T	T	T	T	T
T	F	F	T	T
F	T	T	F	F
F	F	T	T	T

(iii) Truth table for $(q \to \sim p) \leftrightarrow (p \leftrightarrow q)$ is

p	q	$\sim p$	$q \to \sim p$	$p \leftrightarrow q$	$(q \to \sim p) \leftrightarrow$
T	T	F	F	T	
T	F	F	T	F	
F	T	T	T	F	
F	F	T	T	T	

(iv) Truth table for $(p \to q) \leftrightarrow (\sim q \to \sim p)$ is

p	q	$\sim p$	$\sim q$	$p \to q$	$\sim q \to \sim p$	$(p \to$
T	T	F	F	T	T	
T	F	F	T	F	F	
F	T	T	F	T	T	
F	F	T	T	T	T	

(v) Truth table for $\sim (p \vee \sim q) \to \sim p$ is

p	q	$\sim p$	$\sim q$	$p \vee \sim q$	$\sim (p \vee \sim q)$
T	T	F	F	T	F
T	F	F	T	T	F
F	T	T	F	F	T
F	F	T	T	T	F

(vi) Truth table for $((\sim p \to q) \wedge (r \to p)) \leftrightarrow (\sim q \vee r)$ is

p	q	$\sim p$	$\sim q$	$\sim p \to q$	$r \to q$	$\sim q \vee r \equiv a$	$(\sim$
T	T	F	F	T	T	T	
T	T	F	F	T	T	F	
T	F	F	T	T	T	T	
T	F	F	T	T	T	T	
F	T	T	F	T	F	T	
F	T	T	F	T	T	F	
F	F	T	T	F	F	T	
F	F	T	T	F	T	T	

12. Let p : 'I will study Logic and set', q : 'I w... happy' be three propositions. Write each... logical form using the simple propositio...

(i) If I am not happy, then I will go to I... set

(ii) If I am happy, then either I will stu... to USA

(iii) I will not go to USAand I will study... happy.

Also construct the truth table in each c...

Solution (i) The given statement is written a...

(iii) Truth table for $(q \to \sim p) \leftrightarrow (p \leftrightarrow q)$ is

p	q	$\sim p$	$q \to \sim p$	$p \leftrightarrow q$	$(q \to \sim p) \leftrightarrow (p \leftrightarrow q)$
T	T	F	F	T	F
T	F	F	T	F	F
F	T	T	T	F	F
F	F	T	T	T	T

(iv) Truth table for $(p \to q) \leftrightarrow (\sim q \to \sim p)$ is

p	q	$\sim p$	$\sim q$	$p \to q$	$\sim q \to \sim p$	$(p \to q) \leftrightarrow (\sim p \to \sim p)$
T	T	F	F	T	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

(v) Truth table for $\sim(p \vee \sim q) \to \sim p$ is

p	q	$\sim p$	$\sim q$	$p \vee \sim q$	$\sim(p \vee \sim q)$	$\sim(p \vee \sim q) \to \sim p$
T	T	F	F	T	F	T
T	F	F	T	T	F	T
F	T	T	F	F	T	T
F	F	T	T	T	F	T

(vi) Truth table for $((\sim p \to q) \wedge (r \to p)) \leftrightarrow (\sim q \vee r)$ is

p	q	$\sim p$	$\sim q$	$\sim p \to q$	$r \to q$	$\sim q \vee \sim r \equiv a$	$(\sim p \to q) \wedge (r \to p) \equiv b$	$b \to a$
T	T	F	F	T	T	T	T	T
T	T	F	F	T	T	F	T	F
T	F	F	T	T	T	T	T	T
T	F	F	T	T	T	T	T	T
F	T	T	F	T	F	T	F	T
F	T	T	F	T	T	F	T	F
F	F	T	T	F	F	T	F	T
F	F	T	T	F	T	T	F	T

12. **Let p : 'I will study Logic and set', q : 'I will go to USA' and r : 'I am very happy' be three propositions. Write each of the following statements in logical form using the simple propositions p, q and r.**

(i) If I am not happy, then I will go to USA, else I will study Logic and set

(ii) If I am happy, then either I will study Logic and set or I will not go to USA

(iii) I will not go to USAand I will study Logic and set if and only if I am happy.

Also construct the truth table in each case.

Solution (i) The given statement is written as : $(\sim r \to q) \wedge p$

The truth table for $(\sim r \to q) \wedge p$ is

p	q	r	$\sim r$	$\sim r \to q$	$(\sim r \to q) \wedge p$
T	T	T	F	T	T
T	T	F	T	T	T
T	F	T	F	T	F
T	F	F	T	F	F
F	T	T	F	T	F
F	T	F	T	T	F
F	F	T	F	T	F
F	F	F	T	F	F

(ii) The given statement is written as : $(r \to (p \vee \sim q))$

The truth table for $(r \to (p \vee \sim q))$ is

p	q	r	$\sim q$	$p \vee \sim q$	$r \to (p \vee \sim q)$
T	T	T	F	T	T
T	T	F	F	T	T
T	F	T	T	T	T
T	F	F	T	T	T
F	T	T	F	F	T
F	T	F	F	F	T
F	F	T	T	T	T
F	F	F	T	T	T

(iii) The given statement is written as : $(\sim q \wedge p) \Leftrightarrow r$

The truth table for $(\sim q \wedge p) \Leftrightarrow r$ is

p	q	r	$\sim q$	$\sim q \wedge p$	$(\sim q \wedge p) \Leftrightarrow r$
T	T	T	F	F	F
T	T	F	F	F	T
T	F	T	T	T	T
T	F	F	T	T	F
F	T	T	F	F	F
F	T	F	F	F	T
F	F	T	T	F	F
F	F	F	T	F	T

13. **Determine the dual of the following statements**

(i) $p \wedge (q \wedge r)$

(ii) $\sim p \vee \sim q$

(iii) $(p \wedge \sim q) \vee (\sim p \wedge q)$

(iv) $((\sim p \vee q) \wedge (q \wedge \sim r)) \vee (p \vee \mathrm{F})$, **where F is false**

Solution (i) The dual of $p \wedge (q \wedge r)$ is $p \vee (q \vee r)$

(ii) The dual of $\sim p \vee \sim q$ is $\sim p \wedge \sim q$

(iii) The dual of $(p \wedge \sim q) \vee (\sim p \wedge q)$ is $(p \vee \sim q) \wedge (\sim p \vee q)$

(iv) The dual of $((\sim p \vee q) \wedge (q \wedge \sim r)) \vee (p \vee \mathrm{F})$ is $((\sim p \wedge q) \vee (q \vee \sim r)) \wedge (p \wedge \mathrm{T})$

14. **Construct the truth table for each of propositions :**

(i) $(p \to (q \to r)) \to ((p \to q) \to (p \to r))$

(ii) $\sim(p \vee (q \wedge r)) \leftrightarrow ((p \vee q) \wedge (p \to r))$

(iii) $((p \wedge q) \vee \sim(r)) \leftrightarrow p$

(iv) $((p \to q) \to r) \to s$

(v) $(p \leftrightarrow q) \vee (\sim q \leftrightarrow r)$

Solution (i) In the compound proposition $(p \to$ three simple propositions p, q and r are involved. the truth table.

Truth table for $(p \to (q \to r)) \to ((p \to q) \to (p$

p	q	r	$p \to q$	$p \to r$	$q \to r$	$p \to (q \to r) \equiv$
T	T	T	T	T	T	T
T	T	F	T	F	F	F
T	F	T	F	T	T	T
T	F	F	F	F	T	T
F	T	T	T	T	T	T
F	T	F	T	T	F	T
F	F	T	T	T	T	T
F	F	F	T	T	T	T

(ii) Truth table for $\sim(p \vee (q \wedge r)) \leftrightarrow ((p \vee q)$

p	q	r	$q \wedge r$	$p \vee (q \wedge r) \equiv a$	$\sim a$	$p \vee q$
T	T	T	T	T	F	T
T	T	F	F	T	F	T
T	F	T	F	T	F	T
T	F	F	F	T	F	T
F	T	T	T	T	F	T
F	T	F	F	F	T	T
F	F	T	F	F	T	F
F	F	F	F	F	T	F

(iii) Truth table for $((p \wedge q) \vee \sim(r)) \leftrightarrow p$

p	q	r	$\sim r$	$p \wedge q$	
T	T	T	F	T	
T	T	F	T	T	
T	F	T	F	F	
T	F	F	T	F	
F	T	T	F	F	
F	T	F	T	F	
F	F	T	F	F	
F	F	F	T	F	

14. **Construct the truth table for each of the following compound propositions :**

(i) $(p \to (q \to r)) \to ((p \to q) \to (p \to r))$

(ii) $\sim(p \vee (q \wedge r)) \leftrightarrow ((p \vee q) \wedge (p \to r))$

(iii) $((p \wedge q) \vee \sim(r)) \leftrightarrow p$

(iv) $((p \to q) \to r) \to s$

(v) $(p \leftrightarrow q) \vee (\sim q \leftrightarrow r)$ [CU, 2021]

Solution (i) In the compound proposition $(p \to (q \to r)) \to ((p \to q) \to (p \to r))$ three simple propositions p, q and r are involved. So, there must be $2^3 = 8$ rows in the truth table.

Truth table for $(p \to (q \to r)) \to ((p \to q) \to (p \to r))$ is

p	q	r	$p \to q$	$p \to r$	$q \to r$	$p \to (q \to r) \equiv a$	$(p \to q) \to (p \to r) \equiv b$	$a \to b$
T	T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F	T
T	F	T	F	T	T	T	T	T
T	F	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T	T
F	T	F	T	T	F	T	T	T
F	F	T	T	T	T	T	T	T
F	F	F	T	T	T	T	T	T

(ii) Truth table for $\sim(p \vee (q \wedge r)) \leftrightarrow ((p \vee q) \wedge (p \to r))$

p	q	r	$q \wedge r$	$p \vee (q \wedge r) \equiv a$	$\sim a$	$p \vee q$	$p \to r$	$(p \vee q) \wedge (p \to r) \equiv b$	$\sim a \leftrightarrow b$
T	T	T	T	T	F	T	T	T	F
T	T	F	F	T	F	T	F	F	T
T	F	T	F	T	F	T	T	T	F
T	F	F	F	T	F	T	F	F	T
F	T	T	T	T	F	T	T	T	F
F	T	F	F	F	T	T	F	F	T
F	F	T	F	F	T	F	T	F	F
F	F	F	F	F	T	F	T	F	F

(iii) Truth table for $((p \wedge q) \vee \sim(r)) \leftrightarrow p$

p	q	r	$\sim r$	$p \wedge q$	$(p \wedge q) \vee \sim r$	$(p \wedge q) \vee \sim(r)) \leftrightarrow p$
T	T	T	F	T	T	T
T	T	F	T	T	T	T
T	F	T	F	F	F	F
T	F	F	T	F	T	T
F	T	T	F	F	F	F
F	T	F	T	F	T	T
F	F	T	F	F	F	F
F	F	F	T	F	T	F

(iv) In the compound proposition $((p \to q) \to r)$ there are four simple propositions p, q, r and s involved. So, there must be $2^4 = 16$ rows in the truth

p	q	r	s	$p \to q$	$(p \to q) \to r$	$((p \to q) \to r) \to s$
T	T	T	T	T	T	T
T	T	T	F	T	T	F
T	T	F	T	T	F	T
T	T	F	F	T	F	T
T	F	T	T	F	T	T
T	F	T	F	F	T	F
T	F	F	T	F	T	T
T	F	F	F	F	T	F
F	T	T	T	T	T	T
F	T	T	F	T	T	F
F	T	F	T	T	F	T
F	T	F	F	T	F	T
F	F	T	T	T	T	T
F	F	T	F	T	T	F
F	F	F	T	T	F	T
F	F	F	F	T	F	T

Note

Every combination of the truth values of every component is chosen accordi to some rules. In the first column of the truth table corresponding to the fi component, we have taken $\frac{1}{2} \times 2^n$ entries each equal to T, followed by $\frac{1}{2} \times 2$ entries each equal to F. In the second column corresponding to the secon component $\frac{1}{4} \times 2^n$ entries each equal to T have been taken, followed by $\frac{1}{4} \times 2^n$ F's and again followed by $\frac{1}{4} \times 2^n$ T's and again $\frac{1}{4} \times 2^n$ F's. In the third colum corresponding to the third component $\frac{1}{8} \times 2^n$ T's and again $\frac{1}{8} \times 2^n$ F's will be alternatively written starting with T's and so on. Some rule follows with the fourth column corresponding to the fourth component.

(v) Truth table for is $(p \leftrightarrow q) \vee (\sim q \leftrightarrow r)$ is

p	q	r	$\sim q$	$p \leftrightarrow q$	$\sim q \leftrightarrow r$	$(p \leftrightarrow q) \vee (\sim q \leftrightarrow r)$
T	T	T	F	T	F	T
T	T	F	F	T	T	T
T	F	T	T	F	T	T
T	F	F	T	F	F	F
F	T	T	F	F	F	F
F	T	F	F	F	T	T
F	F	T	T	T	T	T
F	F	F	T	T	F	T

15. **The inverse of the statement formula $(\sim r \to$ (i) $(r \to \sim s)$ (ii) $(\sim r \to \sim s)$ (iii) $(s \to r)$ (iv) $(s \to$**

Solution The inverse of $\sim r \to s$ is $\sim(\sim r) \to \sim s \equiv r \to$
Hence, option (i) is correct

16. **Write the converse, inverse and con statements**

(i) If it is raining, then I feel cold

(ii) If it is raining, then I will take coff

(iii) If today is independence day, the

(iv) If it is raining, then the home tea

(v) If it rains, there is cloud in the s

Solution (i) Let us consider the statements
p : It is raining
q : I feel cold
The Converse of the statement is '$q \to$
The Inverse of the statement is '$\sim p \to \sim$
cold.
The Contrapositive of the statement i
is not raining

(ii) Let us consider the statements
p : It is raining
q : I will take coffee
The Converse of the statement is
The Inverse of the statement is '
take coffee
The Contrapositive of the state
it is not raining

(iii) Let us consider the statements
p : Today is Independence day
q : Tomorrow is Thursday
The Converse of the stateme
today is Independence day
The Inverse of the statement
then tomorrow is not Thurs
The Contrapositive of the
Thursday, then today is no

(iv) Let us consider the statem
p : It is raining

15. **The inverse of the statement formula $(\sim r \to s)$ is** [CU, 2021]
(i) $(r \to \sim s)$ (ii) $(\sim r \to \sim s)$ (iii) $(s \to r)$ (iv) $(s \to \sim r)$

Solution The inverse of $\sim r \to s$ is $\sim(\sim r) \to \sim s \equiv r \to \sim s$
Hence, option (i) is correct

16. **Write the converse, inverse and contrapositive of the following statements**
(i) If it is raining, then I feel cold
(ii) If it is raining, then I will take coffee
(iii) If today is independence day, then tomorrow is Thursday
(iv) If it is raining, then the home team wins
(v) If it rains, there is cloud in the sky. [CU, 2021]

Solution (i) Let us consider the statements
p : It is raining
q : I feel cold
The Converse of the statement is '$q \to p$' : If I feel cold, then it is raining
The Inverse of the statement is '$\sim p \to \sim q$' : If it is not raining, then I do not feel cold.
The Contrapositive of the statement is '$\sim q \to \sim p$' : If I do not feel cold, then it is not raining
(ii) Let us consider the statements
p : It is raining
q : I will take coffee
The Converse of the statement is '$q \to p$' : If I take coffee, then it is raining
The Inverse of the statement is '$\sim p \to \sim q$' : If it is not raining, then I will not take coffee
The Contrapositive of the statement '$\sim q \to \sim p$' : If I do not take coffee, then it is not raining
(iii) Let us consider the statements
p : Today is Independence day
q : Tomorrow is Thursday
The Converse of the statement is '$q \to p$' : If tomorrow is Thursday, then today is Independence day
The Inverse of the statement is '$\sim p \to \sim q$' : If today is not Independence day, then tomorrow is not Thursday
The Contrapositive of the statement is '$\sim q \to \sim p$' : If tomorrow is not Thursday, then today is not Independence day
(iv) Let us consider the statements
p : It is raining

q : the home team wins

The Converse of the statement is '$q \to p$' : If the home team wins, then it is raining

The Inverse of the statement is '$\sim p \to \sim q$' : If it is not raining, then the home team does not win

The Contrapositive of the statement '$\sim q \to \sim p$' : If the home team does not win, then it is not raining

(v) Let us consider the statements

p : It rains

q : there is cloud in the sky

The Converse of the statement is '$q \to p$' : If there is cloud in the sky, then it rains

The Inverse of the statement is '$\sim p \to \sim q$' : If it does not rain, then there is no cloud in the sky

The Contrapositive of the statement '$\sim q \to \sim p$' : If there is no cloud in the sky, then it does not rain.

17. **Let $\Sigma = \{a, b, c, d, e\}$. Find (a) $|\Sigma^2|$ and (b) $|\Sigma^3|$**

Solution (a) $|\Sigma^2| = \Sigma\Sigma = 5 \times 5 = 25$

(b) $|\Sigma^3| = 125$

18. **Given the strings $u = a^2bab^2$ and $v = bab^2$, find the string (a) uv, (b) (c) λu and (d) $u\lambda v^2$. Also find their length.**

Solution We write letters of first string following by the letters of the second string

(a) $uv = (a^2bab^2)(bab^2) = a^2bab^3ab^2$; $|uv| = 10$

(b) $vu = (bab^2)(a^2bab^2) = bab^2a^2bab^2$; $|vu| = 10$

(c) $\lambda u = u\lambda = u = a^2bab^2$, since λ is the empty string.

(d) $u\lambda v^2 = uv^2 = (a^2bab^2)(bab^2)(bab^2) = (a^2bab^3ab^2)(bab^2)$
$= a^2bab^3ab^3ab^2$, $|u\lambda v^2| = 14$.

19. **Given the strings $u = a^2ba^3b^2$ and $v = bab^2$, find (a) $|u|$, (b) $|v|$, (c) $|$ and (d) $|v^2|$**

Solution We first count the number of letters in each string and get

(a) $|u| = 8$

(b) $|v| = 4$

(c) $uv = (a^2ba^3b^2)(bab^2) = a^2ba^3b^3ab^2$; $|uv| = 12$

(d) $v^2 = vv = (bab^2)(bab^2) = bab^3ab^2$; $|v^2| = 8$

20. **For the alphabet $\Sigma = \{0,0$**

Solution The set of all strings wh
1's between every pair of 0's.

21. **Let $L_1 = \{a, b\}$ and L_2
Find (a) $(L_1 \cup L_2)^*$ (b) (
(f) $(L_2 - L_1^*)^*$**

Solution (a) $(L_1 \cup L_2)^*$ = set of
(b) $L_1 \cap L_2 = \{b\}$, so $(L_1 \cap L_2$
(c) $L_1^* \cup L_2^*$ = set of all stri
(d) $L_1^* \cap L_2^* = \{b^n : n \geq 0\}$
(e) $L_1 - L_2 = \{a\}$, so $(L_1 - L$
(f) $L_2 - L_1^* = \{c\}$, so $(L_2 -$

22. **Given the strings u**

Solution (a) $uvu = (a^2b^2)($
(b) Since is the empty str

23. **For any strings u**

Solution Let $|u| = r$ and
(a) $|uv| = r + s = |u| + |v|$
(b) $|uv| = |u| + |v| = |v| +$

24. **For the alphabe**
$L_1 = \{0, 1, 00, 1$
$L_2 = \{x : 2 \leq |x|$,
$L_3 = \{x : 2 \geq |x|$,
Determine th
(b) $L_1 \cap L_3$ (c)

Solution (a) $L_1 \cap L_2$
(b) $L_1 \cap L_3 = \{0, 1,$
(c) $L_2 \cap L_3 = \Sigma^*$
(d) $L_1 - L_2 = \{0, 1\}$

20. **For the alphabet $\Sigma = \{0,01\}$, describe the elements of Σ^*.**

Solution The set of all strings which begin and end with 01 and have at least two 1's between every pair of 0's.

21. **Let $L_1 = \{a, b\}$ and $L_2 = \{b, c\}$ be the languages over $\Sigma = \{a, b, c\}$. Find (a) $(L_1 \cup L_2)^*$ (b) $(L_1 \cap L_2)^*$ (c) $L_1^* \cup L_2^*$ (d) $L_1^* \cap L_2^*$ (e) $(L_1 - L_2)^*$ (f) $(L_2 - L_1^*)^*$**

Solution (a) $(L_1 \cup L_2)^*$ = set of all strings in $\Sigma^* = \{a, b, c\}^*$

(b) $L_1 \cap L_2 = \{b\}$, so $(L_1 \cap L_2)^* = \{b^n : n \geq 0\}$

(c) $L_1^* \cup L_2^*$ = set of all strings in Σ^* which are L_1^* or in L_2^*.

(d) $L_1^* \cap L_2^* = \{b^n : n \geq 0\}$

(e) $L_1 - L_2 = \{a\}$, so $(L_1 - L_2)^* = \{a^n : n \geq 0\}$

(f) $L_2 - L_1^* = \{c\}$, so $(L_2 - L_1^*)^* = \{c^n : n \geq 0\}$

22. **Given the strings $u = a^2b^2$ and $v = b^3ab$, find (a) uvu and (b) $u\lambda v$.**

Solution (a) $uvu = (a^2b^2)(b^3ab)(a^2b^2) = (a^2b^5ab)(a^2b^2) = a^2b^5aba^2b^2$

(b) Since is the empty string, $u\lambda v = uv = (a^2b^2)(b^3ab) = a^2b^5ab$.

23. **For any strings u and v, show that (a) $|uv| = |u| + |v|$ and (b) $|uv| = |vu|$**

Solution Let $|u| = r$ and $|v| = s$. Then uv consists of $r + s$ letters. Thus

(a) $|uv| = r + s = |u| + |v|$

(b) $|uv| = |u| + |v| = |v| + |u| = |vu|$

24. **For the alphabet $\Sigma = \{0, 1\}$, the languages L_1, L_2, L_3 are defined as :**

$L_1 = \{0, 1, 00, 11, 000, 111, 0000, 1111\}$

$L_2 = \{x : 2 \leq |x|, x \in \Sigma^*\}$

$L_3 = \{x : 2 \geq |x|, x \in \Sigma^*\}$

Determine the following languages (subsets) of Σ^* (a) $L_1 \cap L_2$ (b) $L_1 \cap L_3$ (c) $L_2 \cap L_3$ and (d) $L_1 - L_2$

Solution (a) $L_1 \cap L_2 = \{00, 11, 000, 111, 0000, 1111\}$

(b) $L_1 \cap L_3 = \{0, 1, 00, 11\}$

(c) $L_2 \cap L_3 = \Sigma^*$

(d) $L_1 - L_2 = \{0, 1\}$

Exercise

A Short and Long Answer Type Questions

1. Which of these sentences are propositions? What are the truth value propositions?
   (i) New Delhi is the capital of India
   (ii) $2 + 6 = 9$
   (iii) $7 + 3 = 10$
   (iv) $x + 7 = 12$
   (v) Find a solution to the problem
   (vi) Who are you?
   (vii) The Sun rotates round the Earth
   (viii) What is your Mother tongue?

2. Consider the following propositions :
   p : Rita is rich
   q : Hema is poor
   Write the following statements in symbolic forms :
   (i) Rita and Hema are both rich
   (ii) Rita is poor and Hema is rich
   (iii) Rita is not rich and Hema is poor
   (iv) Neither Rita nor Hema is poor
   (v) It is not true that Rita and Hema are both rich
   (vi) Either Rita is poor or Hema is poor
   (vii) Either Rita or Hema is poor

3. Consider the following propositions :
   p : He is intelligent
   q : He is brave
   Write the following statements in symbolic forms :
   (i) He is intelligent and brave
   (ii) He is brave or he is not brave and intelligent
   (iii) He is intelligent or not brave

4. Consider the following propositions :
   (i) p : The bag is good
   (ii) q : The bag is cheap
   Write the following statements in symbolic forms :
   (i) The bag is good and cheap
   (ii) The bag is not good but cheap

(iii) This bag is costly but good
(iv) This bag is neither good nor cheap
(v) This bag is good or cheap

5. Consider the following propositions :
p : It is Sunday
q : It is cold
Write the statement against the following symbolic expressions :
(i) $p \vee q$
(ii) $\sim(p \vee q)$
(iii) $\sim p \wedge \sim q$
(iv) $\sim(p \vee \sim q)$

6. Consider the following propositions :
p : Today is Monday
q : It is raining
r : It is cold
Write the statement against the following symbolic expressions :
(i) $p \rightarrow q$
(ii) $\sim p \rightarrow (q \vee r)$
(iii) $p \leftrightarrow q$

7. Consider the following propositions :
p : Today is Saturday
q : It is raining
r : It is cold
Write the statement against the following symbolic expressions :
(i) $p \vee q$
(ii) $\sim p \rightarrow q$
(iii) $p \leftrightarrow r$

8. Consider the following propositions :
p : x is odd number
q : x is divisible by 5
Write the statement against the following symbolic expressions :
(i) $\sim p \wedge \sim q$
(ii) $p \rightarrow q$
(iii) $p \leftrightarrow$

9. Write th negation of the following statements
(i) If he studies, then he will pass in the Examination
(ii) All Negros are tall
(iii) Someone has visited every part of India except New Delhi

(iv) Today is Sunday
(v) No one wants to buy my friend's car
(vi) Every even integer greater than two is not prime
(vii) There is no pollution in West Bengal

10. Write the converse, inverse and contrapositive of the following st
(i) If today is Monday, then I go to the office
(ii) If ABC is a right angled triangle, then $|AB|^2 + |BC|^2 = |AC|^2$
(iii) If n is prime, then n is 2 or n is odd.
(iv) If a triangle is not equilateral, then it is not isosceles
(v) If it is hot, then people will sweat
(vi) If I study hard, then I will pass the Examination

11. Determine the dual of the following statements
(i) $p \wedge (q \vee r)$
(ii) $\sim p \wedge \sim q$
(iii) $(p \wedge \sim q) \wedge (\sim p \wedge q)$
(iv) $((\sim p \vee q) \wedge (q \wedge \sim r)) \vee (p \vee T)$, where T is true

12. Construct the truth table for each of the following compound propos
(i) $(p \wedge q) \rightarrow (p \vee q)$
(ii) $(p \rightarrow q) \leftrightarrow (\sim q \rightarrow \sim p)$
(iii) $(p \rightarrow q) \vee (\sim p \rightarrow q)$
(iv) $(p \rightarrow q) \wedge (\sim p \rightarrow q)$
(v) $(p \leftrightarrow q) \vee (\sim p \leftrightarrow q)$
(vi) $(\sim p \wedge (\sim q \wedge r)) \vee (q \wedge r) \vee (p \wedge r)$
(vii) $(p \rightarrow q) \wedge (q \rightarrow r) \rightarrow (p \rightarrow r)$
(viii) $(p \leftrightarrow q) \vee (\sim q \leftrightarrow r)$
(ix) $((p \rightarrow q) \rightarrow r) \rightarrow s$
(x) $(p \leftrightarrow q) \vee (\sim q \leftrightarrow r)$

13. Let $\Sigma = \{a, b\}$. Describe the following languages over Σ(subsets of Σ^*)
(a) $L_1 = \{(ab)^m : m > 0\}$ (b) $L_2 = \{a^2b^ma^3 : m > 0\}$
(c) $L_3 = \{b^mab^n : m \geq 0, n \geq 0\}$ (d) $L_4 = \{a, ab, ab^2, \ldots\}$

14. For the alphabet $\Sigma = \{0, 1\}$, describe the strings in Σ^* for each o following languages
(a) {01} (b) {000} (c) {0, 010}

15. Let $L_1 = \{a, ab, b^2\}$ and $L_2 = \{b^2, aba\}$ be languages over $\Sigma = \{a, b\}$. F
(a) L_1L_2 (b) L_2L_1 (c) L_2L_2

16. (a) If $L = \{a^2, ba\}$, find L^0, L^2 and L^3.
(b) If $L = \{ab, c\}$ over $\Sigma = \{a, b, c\}$, find L^0, L^3 and L^{-3}.

17. Given $\Sigma = \{a, b, c\}$, find L^* fo
18. Does $(L^2)^* = (L^*)^2$. If not, he
19. Given $u = abcd$. (a) Find initial segments.
20. $A = \{ab, bc, ca\}$, verify whe
(a) abc (b) $ababab$, (c) abb

B Multiple Choice Questions (MC

1. If p : 'Anil is rich' and q statement 'Either Anil o
(a) $p \vee q$ (b) p

2. Let p: It is hot and q: It 'It is hot or it is not rai
(a) $p \vee q$ (b) p

3. If p: It is cold and q: It it is not cold' has the
(a) $\sim q \wedge p$ (b)

4. Let p be a propositio Then the symbolic fo
(a) $p \vee q$ (b) $\sim p \wedge$

5. Let p be a propositio Then $\sim q \wedge \sim p$ states
(a) He is either intell
(c) He is not intellig

6. 'Give me the pen' is
(a) True

7. 'A cat has four legs'
(a) True

8. $5 > 8$ is a proposit
(a) True

9. $x + y = z + x$ is a pr
(a) True

10. Let p : 'It is sunny
(a) It is sunny aft
(b) It is false that
(c) It is not sunn
(d) none of these

17. Given $\Sigma = \{a, b, c\}$, find L^* for (a) $L = \{b^2\}$, (b) $L = \{a, b\}$ and (c) $L = \{a, b, c^3\}$
18. Does $(L^2)^* = (L^*)^2$. If not, how are they related?
19. Given $u = abcd$. (a) Find all subwords of u, and (b) which of them are initial segments.
20. $A = \{ab, bc, ca\}$, verify whether following strings belongs to Σ^*.
    (a) abc (b) $ababab$, (c) $abba$

Multiple Choice Questions (MCQ)

1. **If p : 'Anil is rich' and q : 'Kanchan is poor' then the symbolic from the statement 'Either Anil or Kanchan is rich' is** [MAKAUT 2006]
   (a) $p \vee q$ (b) $p \vee \sim q$ (c) $\sim p \vee q$ (d) $\sim(p \wedge q)$
2. **Let p: It is hot and q: It is raining, then the symbolic form of the statement 'It is hot or it is not raining' is**
   (a) $p \vee q$ (b) $p \wedge q$ (c) $\sim p \vee q$ (d) $p \vee \sim q$
3. **If p: It is cold and q: It is raining, then the statement 'It is not raining and it is not cold' has the symbolic form**
   (a) $\sim q \wedge p$ (b) $\sim q \wedge \sim p$ (c) $\sim(q \wedge p)$ (d) none of these
4. **Let p be a proposition 'He is strong' and q be a proposition 'He is short'. Then the symbolic form of the statement 'He is strong and short' is**
   (a) $p \vee q$ (b) $\sim p \wedge q$ (c) $p \wedge q$ (d) none of these
5. **Let p be a proposition 'He is intelligent' and q be a proposition 'He is tall'. Then $\sim q \wedge \sim p$ states that**
   (a) He is either intelligent or tall. (b) He is neither tall nor intelligent.
   (c) He is not intelligent. (d) He is intelligent and tall.
6. **'Give me the pen' is a proposition**
   (a) True (b) False
7. **'A cat has four legs' is a proposition**
   (a) True (b) False
8. **$5 > 8$ is a proposition**
   (a) True (b) False
9. **$x + y = z + x$ is a proposition**
   (a) True (b) False
10. **Let p: 'It is sunny afternoon' and q: 'It is very hot today'. Then $\sim(p \vee q)$ is**
    (a) It is sunny afternoon or it is very hot today
    (b) It is false that it is sunny afternoon or it is very hot today
    (c) It is not sunny afternoon
    (d) none of these

11. Let p : 'It is sunny afternoon' and q : 'It is hot today'. Then the fo proposition $\sim p \wedge \sim q$ can be written as
(a) It is neither sunny afternoon nor it is hot today
(b) It is false that it is not sunny afternoon or it is hot today
(c) It is false that it is sunny afternoon or it is hot today
(d) none of these

12. Let p and q be the propositions 'She is beautiful' and 'She is respectively. Then the sentence 'It is false that she is beautiful and not fair' is
(a) $\sim p \wedge \sim q$ (b) $\sim(p \vee q)$ (c) $\sim(p \wedge \sim q)$ (d) $q \wedge p$

13. The negation of the statement 'No one wants to buy my apartment'
(a) all want to buy my apartment
(b)some one wants to buy my apartment
(c) every one wants to buy my apartment
(d) none of these

14. If the truth value of the propositions p and q are F and T respectively the truth value of $p \wedge \sim q$ is
(a) F (b) T
(c) both F and T (d) none of these

15. If the truth value of the propositions p, q and r are T, T and F respect then the truth value of $p \wedge (q \vee r)$ is
(a) F (b) T
(c) both F and T (d) none of these

16. For the statement p and q, $\sim(p \vee q)$ implies
(a) $\sim p \wedge \sim q$ (b) $\sim p \vee \sim q$ (c) $\sim p \wedge q$ (d) $p \vee \sim q$

17. $\sim(\sim p \wedge \sim q) \equiv$
(a) $p \vee \sim q$ (b) $p \vee \sim q$ (c) $p \vee q$ (d) none of the

18. $\sim(\sim p \vee \sim q) \equiv$
(a) $p \vee \sim q$ (b) $p \vee \sim q$ (c) $p \wedge q$ (d) none of the

19. $(\sim(\sim p)) \equiv$
(a) p (b) $\sim p$ (c) Tautology (d) Contradict

20. $p \vee (p \wedge q) \equiv$
(a) p (b) q (c) $p \wedge q$ (d) $p \vee q$

21. If the truth value of p and q are T and F respectively then the truth va of $\sim p \to \sim q$ is
(a) F (b) T (c) both F and T (d) none of the

22. $p \to q \equiv$
(a) $\sim q \vee \sim p$ (b) $\sim p \wedge q$ (c) $\sim p \vee q$ (d) $p \vee q$

23. If the truth value of p and q a of $p \to \sim q$ is
(a) F (b) T

24. If the truth value of p and q of $p \leftrightarrow q$ is
(a) T (b) F

25. If $p \leftrightarrow q \equiv (p \to q) \wedge r$, then
(a) $p \to q$ (b) $\sim q$

26. If $f(p, q) = \sim p \wedge \sim q$ then $q = F$ is
(a) F (b) T

27. Converse of is '$\sim p \to q$' is
(a) $p \to \sim q$ (b) q

28. Inverse of '$\sim p \to q$' is
(a) $p \to q$ (b) $\sim p$

29. Contrapositive of '$\sim p \to$
(a) $p \to q$ (b) q

30. If p : Meera is rich, q : Sa "Either Meera or Sara
(a) $p \vee q$ (b)

31. "If X then Y unless Z" propositional logic?
(a) $(X \wedge \sim Z) \to Y$
(c) $X \to (Y \to \sim Z)$

32. P and Q are two equivalent?
(i) $P \vee \sim Q$ (ii) $\sim(\sim P$
(iv) $(P \wedge Q) \vee (P \wedge \sim$
(a) only (i) and (ii)
(c) only (i), (ii) and

33. Which one of the
(a) $(\sim p \vee q) \wedge (p \vee$
(c) $(\sim p \vee q) \vee (p \wedge$

34. The statement ($\sim$
(i) $p \Rightarrow q$ (ii) $q \Rightarrow$
(a) (i) only
(c) (ii) only

23. If the truth value of p and q are F and F respectively then the truth value of $p \to \sim q$ is
(a) F (b) T (c) both T and F (d) none of these

24. If the truth value of p and q are F and T respectively then the truth value of $p \leftrightarrow q$ is
(a) T (b) F (c) both T and F (d) none of these

25. If $p \leftrightarrow q \equiv (p \to q) \wedge r$, then r is
(a) $p \to q$ (b) $\sim q$ (c) $q \to p$ (d) $\sim p$

26. If $f(p, q) = \sim p \wedge \sim q$ then the truth value of f for the truth value p = T, q = F is
(a) F (b) T (c) T or F (d) none of these

27. Converse of is '$\sim p \to q$' is
(a) $p \to \sim q$ (b) $q \to p$ (c) $\sim q \to \sim p$ (d) $q \to \sim p$

28. Inverse of '$\sim p \to q$' is
(a) $p \to q$ (b) $\sim p \to \sim q$ (c) $p \to \sim q$ (d) $\sim q \to \sim p$

29. Contrapositive of '$\sim p \to \sim q$' is
(a) $p \to q$ (b) $q \to \sim p$ (c) $q \to p$ (d) $q \to \sim p$

30. If p: Meera is rich, q: Sara is poor, then the symbolic form of the statement, "Either Meera or Sara is rich" is
(a) $p \vee q$ (b) $p \vee \sim q$ (c) $\sim p \vee q$ (d) $\sim(p \wedge q)$

31. "If X then Y unless Z" is represented by which of the following formulas in propositional logic?
(a) $(X \wedge \sim Z) \to Y$ (b) $(X \wedge Y) \to \sim Z$
(c) $X \to (Y \to \sim Z)$ (d) $(X \to Y) \wedge \sim Z$

32. P and Q are two propositions. Which of the logical expressions are equivalent?
(i) $P \vee \sim Q$ (ii) $\sim(\sim P \wedge Q)$ (iii) $(P \wedge Q) \vee (P \wedge \sim Q) \vee (\sim P \wedge \sim Q)$
(iv) $(P \wedge Q) \vee (P \wedge \sim Q) \vee (\sim P \wedge Q)$
(a) only (i) and (ii) (b) only (i), (ii) and (iii)
(c) only (i), (ii) and (iv) (d) all of these

33. Which one of the following is NOT equivalent to $p \leftrightarrow q$?
(a) $(\sim p \vee q) \wedge (p \vee \sim q)$ (b) $(\sim p \vee q) \vee (q \to q)$
(c) $(\sim p \vee q) \vee (p \wedge \sim q)$ (d) $(\sim p \vee \sim q) \vee (p \wedge q)$

34. The statement $(\sim p) \to (\sim q)$ is logically equivalent to
(i) $p \Rightarrow q$ (ii) $q \Rightarrow p$ (iii) $(\sim q) \vee p$ (iv) $(\sim p) \vee q$
(a) (i) only (b) (i) and (iv) only
(c) (ii) only (d) (ii) and (iii) only

35. $(p \vee \neg p) \wedge q \equiv$
(a) T (b) F (c) p (d) q

36. $(p \vee \neg p) \vee p \equiv$
(a) T (b) F (c) p (d) q

37. $p \rightarrow (q \wedge \neg q) \equiv$
(a) $\neg p$ (b) $\neg q$ (c) F (d) T

38. $\neg(p \leftrightarrow q) \equiv$
(a) $(p \wedge q) \vee (\neg p \wedge \neg q)$ (b) $(p \wedge q) \wedge (\neg p \wedge \neg q)$
(c) $(p \wedge \neg q) \vee (\neg p \wedge q)$ (d) $(p \vee \neg q) \wedge (\neg p \wedge q)$

39. $p \rightarrow (q \wedge q)$ is
(a) $p \rightarrow q$ (b) $q \rightarrow p$ (c) p (d) q

40. The negations of $\sim(\sim p) \equiv$
(a) $\sim p$ (b) p (c) $\sim(\sim p)$ (d) none

41. The inverse of the statement formula $(\sim r \rightarrow s)$ is
(a) $(r \rightarrow \sim s)$ (b) $(\sim r \rightarrow \sim s)$ (c) $(s \rightarrow r)$ (d) $(s \rightarrow \sim r)$

Answers

A. Short and Long Answer Type Questions

1. (i) Yes, T (ii) Yes, F (iii) Yes, T (iv) No (v) No (vi) No (vii) Yes, F (viii)
2. (i) $p \wedge \sim q$ (ii) $\sim p \wedge \sim q$ (iii) $\sim p \wedge q$ (iv) $p \wedge \sim q$ (v) $\sim(p \wedge \sim q)$
(vi) $\sim p \vee q$ (vii) $\sim p \vee q$
3. (i) $p \wedge q$ (ii) $q \vee (\sim q \wedge p)$ (iii) $p \vee \sim q$
4. (i) $p \wedge q$ (ii) $\sim p \wedge q$ (iii) $\sim q \wedge p$ (iv) $\sim p \wedge \sim q$ (v) $p \vee q$
5. (i) It is Sunday or it is cold
(ii) It is not true that it is Sunday or it is cold
(iii) It is neither Sunday nor it is cold
(iv) It is not true that it is Sunday or it is not cold.
6. (i) If today is Monday, then it is raining
(ii) If today is not Monday, then it is raining or it is cold
(iii) Today is Monday if and only if it is raining
7. (i) Today is Saturday or it is raining
(ii) If today is not Saturday, then it is raining
(iii) Today is Saturday if and only if it is cold
8. (i) x is not odd number and x is not divisible by 5
(ii) If x is odd number, then x is divisible by 5
(iii) x is odd number if and only if x is divisible by 5

9. (i) If he does not study, then he will not pass in the examination.
(ii) Some Nigros are short
(iii) Everyone has either visited New Delhi or has not visited any part of India other than New Delhi
(iv) Today is not Sunday
(v) Someone wants to buy my friend's car
(vi) Some even integer greater than two is prime
(vii) There pollution in West Bengal

10. (i) **Converse :** If I go to office, then today is Monday
Inverse : If today is not Monday, then I do not go to office
Contrapositive : If I do not go to office, then today is not Monday

(ii) **Converse :** If $|AB|^2 + |BC|^2 = |AC|^2$, then the triangle ABC is a right angled triangle.
Inverse : If ABC is not a right angled triangle, then $|AB|^2 + |BC|^2 \neq |AC|^2$
Contrapositive : If $|AB|^2 + |BC|^2 \neq |AC|^2$, then the triangle ABC is not a right angled triangle.

(iii) **Converse :** If n is odd or n is 2, then n is prime
Inverse : If n is not prime, then n is not odd or n is not 2
Contrapositive : If n is not odd or not 2, then n is not prime

(iv) **Converse :** If a triangle is not isosceles, then it is not equilateral
Inverse : If a triangle is equilateral, then it is isosceles
Contrapositive : If a triangle is isosceles, then it is equilateral

(v) **Converse :** If people will sweat, then it is hot
Inverse : If it is not hot, then people will not sweat
Contrapositive : If people will not sweat, then it is not hot

(vi) **Converse :** If I pass in the Examination, then I will study hard
Inverse : If I do not study hard, then I will not pass in the Examination
Contrapositive : If I do not pass in the Examination, then I will not study hard

11. (i) $p \vee (q \wedge r)$ (ii) $\sim p \vee \sim q$ (iii) $(p \vee \sim q) \vee (\sim p \vee q)$
(iv) $((\sim p \wedge q) \vee (q \vee \sim r)) \wedge (p \wedge F)$, where F is False.

12. (i) T T T T (ii) T T T T (iii) T T T T (iv) T T T F (v) T T T T (vi) T F T F T F T T
(vii) The given statement is a tautology (all T) (viii) T T T F F T T T

13. (a) L_1 consists of all strings that begin with a, alternating with b and ending with b.
(b) L_2 consists of all strings that begin with a^2, end with a^3 with one or more b's between them.
(c) L_3 consists of all strings with exactly one a.
(d) L_4 consists of all strings that begin with a and followed by 0's or more b's.

14. (a) Σ^* consists of all strings x of even length. If $x \neq \lambda$, then x begins with ... ends with 1 and the symbol (0 and 1) alternate.

(b) Σ^* consists of those strings that are made up of $3n$ 0's for $n \in N$.

(c) Any string $x \in \Sigma^*$ provided (i) x is a string of n 0's for $n \in N$ or (ii) ... string that starts and ends with 0 and has at least one 1 and at le... 0's between any two 1's.

15. (a) Concatenate words in L_1 with words in L_2 to obtain

$L_1L_2 = \{a, ab, b^2\}\{b^2, aba\} = \{ab^2, a^2ba, ab^3, ababa, b^4, b^2aba\}$

$L_2L_1 = \{b^2, aba\}\{a, ab, b^2\} = \{b^2a, a^2ba, b^2ab, a^2b^2a, abab^2\}$

$L_2L_2 = \{b^2, aba\}\{b^2, aba\} = \{b^4, b^2aba, abab^2, aba^2ba\}$

16. (a) (i) $L^0 = \{\lambda\}$ by definition

(ii) All two-word sequences from L

$L^2 = \{a^2, ba\}\{a^2, ba\} = \{a^4, a^2ba, ba^3, baba\}$

(iii) L^3 = All three-word sequences from L

$= \{a^6, a^4ba, a^2ba^3, a^3baba, ba^5, ba^2, ba^2ba, baba^3, bababa\}$

(b) (i) $L^0 = \{\lambda\}$

(ii) $L^3 = \{ababab, ababc, abacab, abc^2, cabab, cabc, c^2, ab, c^3\}$

(ii) The negative power of a language is not defined.

17. (a) $L^* = \{b^n : n \text{ is even number}\}$

(b) $L^* = \{a, b\}$

(c) L^* consists of all words from Σ and the length of each maximal subwo... are composed of e's divisible by 3.

18. Not true, because $(L)^2 \subseteq (L^*)^2$

19. (a) Subwords of u are : λ, a, b, c, d, ab, bc, cd, abc, bcd, $abcd$. Here the wo... not the subword of u even though all letters belong to u.

(b) The initial segments are : λ, a, ab, abc, $abcd$.

20. (a) No (b) yes (c) No (d) No

B. Multiple Choice Questions (MCQ)

1. b	2. d	3. b	4. c	5. b	6. b	7. a	8. a	9. b	10.
11. a	12. c	13. b	14. a	15. b	16. a	17. c	18. c	19. a	20.
21. b	22. c	23. a	24. b	25. c	26. a	27. d	28. c	29. c	30.
31. a	32. a	33. c	34. d	35. d	36. a	37. a	38. c	39. a	40.
41. a									

02

Propositio...

Introduction

In the previous chapter we studied ... truth values. In this chapter we will di... can be linked to form a logically vali... with propositions is known as propo... The language of a propositional logi... set of logical operators. An expressio... natural language (English, Bengali etc... a propositional logic is to formalize ...

An argument that establishes the ... proof and logic is a tool for the analys... it is especially concerned with whet... about different theorems and abou... are statements that are provided to... kinds of proofs.

A proof uses the hypothesis, ... conclusion. Each step of proof in... whole to be valid, each step of the... chapter we shall discuss the 'Rule... used within larger arguments or ... the justification of the steps used... a set of hypothesis. The most ba... sentences, expressed by such fre... combinations allow us to describ... have or lack: something is 'not ... 'classification', and it is the ba... time, these logical sentence ... sense, as they structure how ... When we disagree with a cla...

02

Propositional Logic

Introduction

In the previous chapter we studied about propositional statements and their truth values. In this chapter we will discuss about the way in which statements can be linked to form a logically valid argument. The area of logic that deals with propositions is known as propositional logic or, propositional calculus. The language of a propositional logic consists of propositional variables and set of logical operators. An expression in a propositional logic is expressed as a natural language (English, Bengali etc.) or by symbols and variables. The goal of a propositional logic is to formalize reasoning with logical operators.

An argument that establishes the truth of a statement or theorem is called a proof and logic is a tool for the analysis of proofs. As logic is the study of reasoning, it is especially concerned with whether the reasoning is correct. We shall discuss about different theorems and about mathematically accepted proofs. Theorems are statements that are provided to be true. Lemmas and Corollaries are special kinds of proofs.

A proof uses the hypothesis, axioms, definitions and so on, to reach a conclusion. Each step of proof involves drawing conclusions. For the proof as a whole to be valid, each step of the proof must result in a valid conclusion. In this chapter we shall discuss the 'Rules of Inference' which are brief valid arguments used within larger arguments or proofs to test their validity. These rules provide the justification of the steps used to show that a conclusion follows logically from a set of hypothesis. The most basic logical inferences are about combinations of sentences, expressed by such frequent expressions as 'not', 'and', 'or', 'if then'. Such combinations allow us to describe situations, and what properties these situations have or lack: something is 'not this, but that'. We could call this reasoning about 'classification', and it is the basis of any description of the world. At the same time, these logical sentence combinations are also fundamental in another sense, as they structure how we communicate and engage in argumentation. When we disagree with a claim that someone makes, we often try to derive a

consequence ('if then') whose negation ('not') is easier to show. We will
all these patterns of reasoning below. More precisely, in this first chapte
shall introduce propositional logic, the logical system behind the reaso
with 'not', 'and', 'or', 'if then' and other basic sentence-combining operato
will get acquainted with the notions of formula, logical connective, truth
consequence, information update, formal proof, and expressive power, whil
also present some backgrounds in computation and cognition

2.1 Syntax and Semantics of Propositional Logic

In chapter 1 we have already discussed that deductive reasoning is a lo
reasoning in which a conclusion is based on the concordance of mu
premises that are generally assumed to be true. The branch of logic concer
exclusively with the principles of deductive reasoning and with the form ra
than the content of propositions is **Formal logic**.

So, Formal logic is the study of inferences (logical conclusions) where
premises must be true. **Informal logic** is the reasoning and arguments we m
in our personal exchanges with others. It is a theorical alternative to formal lo

In Formal logic syntax refers to the formal notation for writing statements t
are either true or false. On the other hand, semantics is the meaning of a form
in terms of mathematical entities. Syntax refers to grammar, while seman
refers to meaning. Syntax is the set of rules needed to ensure a sentenc
grammatically correct; semantics is how one's lexicon, grammatical structu
tone, and other elements of a sentence in a particular language combine
communicate its meaning. For example, at the concept of syntax, a string of th
symbols is 5 + 7. But, the strings say 5 + 7 and 7 + 5 are syntactically different b
they are of same sense in terms of semantics. The semantics of a propositio
statement is either true or false.

2.2 Tautology

Tautology is a propositional statement form in which all the substitut
instances must be true. Hence, a propositional statement P = $f(p, q, r, \ldots)$
known to be a tautology if it is true for every truth assignment for $p, q, r, \ldots$ T
means that in the truth table of the formula every entry in the last column w
be True (T). This explains that a statement form is a tautology if and only if
corresponding truth function takes only the value T, or equivalently, if, in its tru
table, the column under the statement form contains only Ts.

Example 2.1 : Let us consider the propositional statement $f(p, q) = p \rightarrow (p \vee$

The truth table of a propositional statement $f(p, q) = p \rightarrow (p \vee q)$ is

[KU, 201

p	q	
T	T	
T	F	
F	T	
F	F	

We find that all the entries in
propositional Formula f $(p, q) = p \rightarrow$
So, $f(p, q) = p \rightarrow (p \vee q)$ is a taut

2.3 Contradictions

Contradiction is a propositio
instances are false. Hence, a prop
be a Contradiction or selfcontradi
$p, q, r, \ldots$. This means that in the t
column will be False (F). This me
and only if its corresponding truth
if, in its truth table, the column

Example 2.2 : Let us conside

$f(p, q) = (p \wedge q) \wedge \sim (p \vee$

The truth table of a Proposit

p	q	p∧q	
T	T	T	
T	F	F	
F	T	F	
F	F	F	

We find that all the entri
propositional formula $f(p, q)$
So, $f(p, q) = (p \wedge q) \wedge \sim (p \vee q)$

Notes

1. The negation of a Tau
contradiction is a tautol
2. The propositional Staten
among their substitutio
Any statement whose
statement. Thus, if a pr
tautology nor a contrad

Table 2.1

p	q	$p \vee q$	$p \rightarrow (p \vee q)$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	T

We find that all the entries in the last column of the truth table for the propositional Formula f $(p, q) = p \rightarrow (p \vee q)$ assumes only truth values (T).

So, $f(p, q) = p \rightarrow (p \vee q)$ is a tautology.

2.3 Contradictions

Contradiction is a propositional statement form all of whose substitution instances are false. Hence, a propositional Formula P = $f(p, q, r, ...)$ is known to be a Contradiction or selfcontradictory, if it is False for every truth assignment for $p, q, r, ...$ This means that in the truth table of the formula every entry in the last column will be False (F). This means that a statement form is a Contradiction if and only if its corresponding truth function takes only the value F, or equivalently, if, in its truth table, the column under the statement form contains only Fs.

Example 2.2 : Let us consider the Propositional Formula

$f(p, q) = (p \wedge q) \wedge \sim (p \vee q)$

The truth table of a Propositional Formula $f(p, q) = (p \wedge q) \wedge \sim (p \vee q)$ is

Table 2.2

p	q	$p \wedge q$	$p \vee q$	$\sim (p \vee q)$	$(p \wedge q) \wedge \sim (p \vee q)$
T	T	T	T	F	F
T	F	F	T	F	F
F	T	F	T	F	F
F	F	F	F	T	F

We find that all the entries in the last column of the truth table for the propositional formula $f(p, q) = (p \wedge q) \wedge \sim (p \vee q)$ assumes only False (F) values.

So, $f(p, q) = (p \wedge q) \wedge \sim (p \vee q)$ is a Contradiction.

Notes

1. The negation of a Tautology is a contraction and the negation of a contradiction is a tautology.
2. The propositional Statement forms that have both true and false statements among their substitution instances are called ***contingent statement forms.*** Any statement whose specific form is contingent is called a ***contingent statement.*** Thus, if a propositional Statement P = $f(p, q, r, ...)$ is neither a tautology nor a contradiction, it is called a ***Contingency.***

2.4 Logical Equivalence

We now introduce a new relation, important and very useful which is n connective but is a powerful concept in Propositional Calculus. Two compo propositions $P = f_1(p, q, r, ...)$ and $Q = f_2(p, q, r, ...)$ are said to be logically equiva or simply equivalent, if they have identical truth values, viz. if the truth valu P is equal to the truth value of Q for every one of the 2^n possible sets of t values assigned for $p, q, r, ...$ This equivalence is denoted by $P \equiv Q$ (read as equivalent to Q).

By means of a truth table, we have an effective procedure for determi whether a statement form is a tautology. Truth table gives us an effect procedures for determining whether a given statement form logically impl another given statement form and whether two given statement forms logically equivalent.

A statement (in some natural language like English or in a formal theory) t arises from a tautology by the substitution of statements for all the stateme letters, with occurrences of the same statement letter being replaced by t same statement, is said to be *logically true*. Thus, a logically true stateme is a statement which is true regardless of the truth or falsity of its constitue propositions. In other words, it is a statement which is not only true, but o which is true under all interpretations of its logical components other than l logical constants. Logical truth is a very important concept in propositiona calculus. Let us consider an example in the natural language English:

'If he is rich or he is emotional, and he is not emotional, then he is rich', whic arises by substitution from the tautology $(((p \vee q) \wedge (\sim q)) \rightarrow p)$

Notes

1. A statement that is formed from a contradictory statement form by means of substitution is said to be *logically false*
2. The symbol ≡ is not a connective.

Example 2.3 : Let us consider the propositional statements $p \rightarrow q$ and $\sim p \vee$

The truth tables of the statements $p \rightarrow q$ and $\sim p \vee q$ are

Table 2.3

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

and

Table 2.4

p	q	$\sim p$	$\sim p \vee q$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

We find that the final columns in the truth tables of the statements $p \rightarrow q$ and $\sim p \vee q$ are identical.

Hence, $p \rightarrow q \equiv \sim p \vee q$.

2.5 Principal of Duality

The dual of a compound proposition ∨, ∧ and ~ is the proposition obtained each T by F and each F by T, where compound propositions that are tau dual of the proposition P is denoted

If $P(p, q, r, ...) \equiv Q(p, q, r, ...)$, whe $P^*(p, q, r, ...) \equiv Q^*(p, q, r, ...)$.

Example 2.4 : The dual of $p \vee$

2.6 Algebra of Propo

For any propositions p, q, r th Algebra as shown in the following

Table 2.5 : Law

Sl. No.	Name of law	
1	Idempotent law	$p \vee$
2	Identity law	$p \vee$
3	Dominant law	$p \vee$
4	Complement law	$p \vee$
5	Commutative law	$p \vee$
6	Associative law	$(p$
7	Distributive law	p
8	Absorption law	p
9	De Morgan's law	$\sim$

For any propositions p, shown in the following table

Table 2.6 : D

Sl. No.	
1	p
2	p
3	p
4	
5	
6	
7	
8	
9	

2.5 Principal of Duality

The dual of a compound proposition that contains only the logical operators $\vee$, $\wedge$ and $\sim$ is the proposition obtained by replacing each $\vee$ by $\wedge$, each $\wedge$ by $\vee$, each T by F and each F by T, where T and F are special variables representing compound propositions that are tautologies and contradictions respectively. The dual of the proposition P is denoted by P^*.

If $P(p, q, r, \ldots) \equiv Q(p, q, r, \ldots)$, where P and Q are compound propositions, then $P^*(p, q, r, \ldots) \equiv Q^*(p, q, r, \ldots)$.

Example 2.4 : The dual of $p \vee (q \vee r)$ is $p \wedge (q \wedge r)$

2.6 Algebra of Propositions

For any propositions p, q, r the logical connectives $\wedge$, $\vee$, $\sim$ obey the laws of Algebra as shown in the following table

Table 2.5 : Laws of Algebra of Propositions

Sl. No.	Name of law	Primal form	Dual form
1	Idempotent law	$p \vee p \equiv p$	$p \wedge p \equiv p$
2	Identity law	$p \vee F \equiv p$	$p \wedge T \equiv p$
3	Dominant law	$p \vee T \equiv T$	$p \wedge F \equiv F$
4	Complement law	$p \vee \sim p \equiv T$	$p \wedge \sim p \equiv F$
5	Commutative law	$p \vee q \equiv q \vee p$	$p \wedge q \equiv q \wedge p$
6	Associative law	$(p \vee q) \vee r \equiv p \vee (q \vee r)$	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
7	Distributive law	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
8	Absorption law	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
9	De Morgan's law	$\sim(p \vee q) \equiv \sim p \wedge \sim q$	$\sim(p \wedge q) \equiv \sim p \vee \sim q$

For any propositions p, q, r the equivalences involving conditional '$\rightarrow$' are shown in the following table

Table 2.6 : Equivalences involving Conditionals

Sl. No.	Equivalences
1	$p \rightarrow q \equiv \sim p \vee q$
2	$p \rightarrow q \equiv \sim q \rightarrow \sim p$
3	$p \vee q \equiv \sim p \rightarrow q$
4	$p \wedge q \equiv \sim(p \rightarrow \sim q)$
5	$\sim(p \rightarrow q) \equiv p \wedge \sim q$
6	$(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$
7	$(p \rightarrow r) \wedge (q \rightarrow r) \equiv (p \vee q) \rightarrow r$
8	$(p \rightarrow q) \vee (p \rightarrow r) \equiv p \rightarrow (q \vee r)$
9	$(p \rightarrow r) \vee (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$

For any propositions p, q, r the equivalences involving Biconditional are shown in the following table

Table 2.7: Equivalences involving Biconditional

Sl. No.	Equivalences
1	$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$
2	$p \leftrightarrow q \equiv \sim p \leftrightarrow \sim q$
3	$p \leftrightarrow q \equiv (p \wedge q) \vee (\sim p \wedge \sim q)$
4	$\sim (p \leftrightarrow q) \equiv p \leftrightarrow \sim q$

2.7 Truth tables for Equivalences

In this Section we shall verify some laws of Algebra, equivalences involving Conditional and Biconditional using truth tables

Example 2.5 : Using truth table we shall verify the distributive law

$p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$

Table 2.8 : $p \wedge (q \vee r)$

p	q	r	$q \vee r$	$p \wedge (q \vee r)$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	F	F
F	T	T	T	F
F	T	F	T	F
F	F	T	T	F
F	F	F	F	F

Table 2.9 : $(p \wedge q) \vee (p \wedge r)$

p	q	r	$p \wedge q$	$p \wedge r$	$(p \wedge q) \vee (p \wedge r)$
T	T	T	T	T	T
T	T	F	T	F	T
T	F	T	F	T	T
T	F	F	F	F	F
F	T	T	F	F	F
F	T	F	F	F	F
F	F	T	F	F	F
F	F	F	F	F	F

From the above tables we see that the two propositional statements $p \wedge (q \vee r)$ and $(p \wedge q) \vee (p \wedge r)$ have the same truth tables.

Hence, $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

Example 2.6 : Using truth table we shall verify the De Morgan's law

$$\sim(p \vee q) \equiv \sim p \wedge \sim q$$

Truth table for $\sim(p \vee q)$ is

Table 2.10 : $\sim(p \vee q)$

p	q	$p \vee q$	$\sim(p \vee q)$
T	T	T	F
T	F	T	F
F	T	T	F
F	F	F	T

Table 2.11 : $\sim p \wedge \sim q$

p	q	$\sim p$	$\sim q$	$\sim p \wedge \sim q$
T	T	F	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

From the above tables we see that the two propositional statements $\sim(p \vee q)$ and $\sim p \wedge \sim q$ have the same truth tables.

Hence, $\sim(p \vee q) \equiv \sim p \wedge \sim q$

Example 2.7 : Using truth table we shall verify $p \rightarrow q \equiv \sim p \vee q$

Table 2.12 : $p \rightarrow q$

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Table 2.13 : $\sim p \vee q$

p	q	$\sim p$	$\sim p \vee q$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

From the above tables we see that the two propositional statements $p \rightarrow q$ and $\sim p \vee q$ have the same truth tables. Hence, $p \rightarrow q \equiv \sim p \vee q$

Example 2.8 : Using truth table we shall verify $(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$

Table 2.14 : $(p \rightarrow q) \wedge (p \rightarrow r)$

p	q	r	$p \rightarrow q$	$p \rightarrow r$	$(p \rightarrow q) \wedge (p \rightarrow r)$
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	F	T	F
T	F	F	F	F	F
F	T	T	T	T	T
F	T	F	T	T	T
F	F	T	T	T	T
F	F	F	T	T	T

Table 2.15 : $p \to (q \wedge r)$

p	q	r	$q \wedge r$	$p \to (q \wedge r)$
T	T	T	T	T
T	T	F	F	F
T	F	T	F	F
T	F	F	F	F
F	T	T	T	T
F	T	F	F	T
F	F	T	F	T
F	F	F	F	T

From the above tables we see that the two propositional statements $(p \to q) \wedge (p \to r)$ and $p \to (q \wedge r)$ have the same truth tables.

Hence, $(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$

Example 2.9 : Using truth table we shall verify $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$

Table 2.16 : $p \leftrightarrow q$

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Table 2.17 : $(p \to q) \wedge (q \to p)$

p	q	$p \to q$	$q \to p$	$(p \to q) \wedge (q \to p)$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

From the above tables we see that the two propositional statements $p \leftrightarrow q$ and $(p \to q) \wedge (q \to p)$ have the same truth tables. Hence, $p \leftrightarrow q \equiv (p \to q) \wedge (q \to$

2.8 Laws of Algebra of Propositions for Equivalence

When a statement formula involves a large number of simple propositions then using truth table may be too complex in proving logical equivalence. To make the proves of equivalence a simpler one, the use of laws of Algebra is more fruitful in the real scenario.

We shall verify some equivalence using Laws of Algebra of Propositions.

Example 2.10 : Prove the following equivalences without using truth table.

(i) $p \vee (p \wedge q) \equiv p$

(ii) $p \to (q \to r) \equiv (p \wedge q) \to r$

(iii) $p \leftrightarrow q \equiv (p \vee q) \to (p \wedge q)$

(iv) $(\sim p \wedge (\sim q \wedge r)) \vee (q \wedge r) \vee (p \wedge r) \equiv$

(v) $((p \vee \sim q) \wedge (\sim p \vee \sim q)) \vee q \equiv T$ (Tau

(vi) $(p \vee q) \wedge (\sim p \wedge \sim q) \equiv F$ (Contradict

(i) Here, $p \vee (p \wedge q) \equiv (p \wedge T) \vee (p \wedge$
$\equiv p \wedge (T \vee q)$ [by
$\equiv p \wedge T$ [by Dom
$\equiv p$ [by Identity
Thus, $p \vee (p \wedge q) \equiv p$

(ii) Here, $p \to (q \to r) \equiv p \to (\sim q \vee r)$
$\equiv \sim p \vee (\sim q \vee r$
$\equiv (\sim p \vee \sim q) \vee$
$\equiv \sim (p \wedge q) \vee r$
$\equiv (p \wedge q) \to$
Thus, $p \to (q \to r) \equiv (p \wedge q) \to$

(iii) $p \leftrightarrow q \equiv (p \vee q) \to (p \wedge q)$
Here, $p \leftrightarrow q \equiv (p \to q) \wedge (q \to$
[Bicondi
$\equiv (\sim p \vee q) \wedge (\sim q$
$\equiv ((\sim p \vee q) \wedge \sim q$
$\equiv (\sim q \wedge (\sim p \vee q)$
$\equiv ((\sim q \wedge \sim p) \vee$
$\equiv ((\sim q \wedge \sim p) \vee$
$\equiv (\sim q \wedge \sim p) \vee$
$\equiv \sim (q \vee p) \vee (p$
$\equiv \sim (p \vee q) \vee (p$
$\equiv (p \vee q) \to (p$
Thus, $p \leftrightarrow q \equiv (p \to q) \wedge$

(iv) $(\sim p \wedge (\sim q \wedge r)) \vee (q \wedge r) \vee$
Here, $(\sim p \wedge (\sim q \wedge r)) \vee (q$
$\equiv ((\sim p \wedge \sim q) \wedge$
$\equiv ((\sim p \wedge \sim q)$
$\equiv (\sim (p \vee q) \vee$
$\equiv (\sim (p \vee q) \vee$
$\equiv T \wedge r$ [Con
$\equiv r$ [Identity
Thus, $(\sim p \wedge (\sim q \wedge r)) \vee (q$

(iv) $(\sim p \wedge (\sim q \wedge r)) \vee (q \wedge r) \vee (p \wedge r) \equiv r$

(v) $((p \vee \sim q) \wedge (\sim p \vee \sim q)) \vee q \equiv \mathrm{T}$ (Tautology)

(vi) $(p \vee q) \wedge (\sim p \wedge \sim q) \equiv \mathrm{F}$ (Contradiction)

(i) Here, $p \vee (p \wedge q) \equiv (p \wedge \mathrm{T}) \vee (p \wedge q)$ [by Identity law, $p \wedge \mathrm{T} \equiv p$]

$\equiv p \wedge (\mathrm{T} \vee q)$ [by distributive law]

$\equiv p \wedge \mathrm{T}$ [by Dominant law, $\mathrm{T} \vee q \equiv \mathrm{T}$]

$\equiv p$ [by Identity law, $p \wedge \mathrm{T} \equiv p$]

Thus, $p \vee (p \wedge q) \equiv p$

(ii) Here, $p \to (q \to r) \equiv p \to (\sim q \vee r)$ [Conditional equivalence, $q \to r \equiv \sim q \vee r$]

$\equiv \sim p \vee (\sim q \vee r)$ [Conditional equivalence]

$\equiv (\sim p \vee \sim q) \vee r$ [Associative law]

$\equiv \sim(p \wedge q) \vee r$ [De Morgan's law]

$\equiv (p \wedge q) \to r$ [Conditional equivalence]

Thus, $p \to (q \to r) \equiv (p \wedge q) \to r$

(iii) $p \leftrightarrow q \equiv (p \vee q) \to (p \wedge q)$

Here, $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$

[Biconditional equivalence $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$]

$\equiv (\sim p \vee q) \wedge (\sim q \vee p)$ [Conditional equivalence]

$\equiv ((\sim p \vee q) \wedge \sim q) \vee ((\sim p \vee q) \wedge p)$ [Distributive law]

$\equiv (\sim q \wedge (\sim p \vee q)) \vee (p \wedge (\sim p \vee q))$ [Commutative law]

$\equiv ((\sim q \wedge \sim p) \vee (\sim q \wedge q)) \vee ((p \wedge \sim p) \vee (p \wedge q))$ [Distributive law]

$\equiv ((\sim q \wedge \sim p) \vee \mathrm{F}) \vee (\mathrm{F} \vee (p \wedge q))$ [Complement law]

$\equiv (\sim q \wedge \sim p) \vee (p \wedge q)$ [Identity law]

$\equiv \sim(q \vee p) \vee (p \wedge q)$ [De Morgan's law]

$\equiv \sim(p \vee q) \vee (p \wedge q)$ [Commutative law]

$\equiv (p \vee q) \to (p \wedge q)$ [Conditional equivalence]

Thus, $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$

(iv) $(\sim p \wedge (\sim q \wedge r)) \vee (q \wedge r) \vee (p \wedge r) \equiv r$

Here, $(\sim p \wedge (\sim q \wedge r)) \vee (q \wedge r) \vee (p \wedge r) \equiv (\sim p \wedge (\sim q \wedge r)) \vee ((q \vee p) \wedge r)$

[Distributive law]

$\equiv ((\sim p \wedge \sim q) \wedge r) \vee ((q \vee p) \wedge r)$ [Associative law]

$\equiv ((\sim p \wedge \sim q) \vee (q \vee p)) \wedge r$ [Distributive law]

$\equiv (\sim(p \vee q) \vee (q \vee p)) \wedge r$ [De Morgan's law]

$\equiv (\sim(p \vee q) \vee ((p \vee q)) \wedge r$ [Commutative law]

$\equiv \mathrm{T} \wedge r$ [Complement law]

$\equiv r$ [Identity law]

Thus, $(\sim p \wedge (\sim q \wedge r)) \vee (q \wedge r) \vee (p \wedge r) \equiv r$

(v) $((p \vee \sim q) \wedge (\sim p \vee \sim q)) \vee q \equiv T$ (Tautology)

Here, $((p \vee \sim q) \wedge (\sim p \vee \sim q)) \vee q$

$\equiv ((p \vee \sim q) \wedge \sim p) \vee ((p \vee \sim q) \wedge \sim q)) \vee q$ [Distributive law]

$\equiv (((p \wedge \sim p) \vee (\sim q \wedge \sim p)) \vee ((p \wedge \sim q) \vee (\sim q \wedge \sim q))) \vee q$ [Distribu...

$\equiv ((F \vee (\sim q \wedge \sim p)) \vee ((p \wedge \sim q) \vee (\sim q)) \vee q$

[Complement law and Idempote...

$\equiv (\sim q \wedge \sim p)) \vee ((p \wedge \sim q) \vee (\sim q)) \vee q$ [Identity law]

$\equiv (\sim q \wedge \sim p) \vee (p \wedge \sim q) \vee (\sim q \vee q)$ [Associative law]

$\equiv (\sim q \wedge \sim p) \vee (p \wedge \sim q) \vee T$ [Complement law]

$\equiv (\sim q \wedge \sim p) \vee T$ [Dominant law]

$\equiv T$ [Dominant law]

Thus, $((p \vee \sim q) \wedge (\sim p \vee \sim q)) \vee q \equiv T$

(vi) $(p \vee q) \wedge (\sim p \wedge \sim q) \equiv F$ (Contradiction)

Here, $(p \vee q) \wedge (\sim p \wedge \sim q) \equiv (p \vee q) \wedge \sim (p \vee q)$ [De Morgan's law]

$\equiv F$ [Complement law]

Thus, $(p \vee q) \wedge (\sim p \vee \sim q) \equiv F$

Illustrative Examples

1. **Let Determine which of the following compound propositions ... tautologies and which of them are contradictions using truth table**

(i) $(\sim q \wedge (p \to q)) \to \sim p$

(ii) $\sim p \to (p \to q)$

(iii) $((p \to q) \wedge (q \to r)) \to (p \to r)$

(iv) $(p \vee q) \wedge (\sim p \wedge \sim q)$

(v) $\sim (q \to r) \wedge r \wedge (p \to q)$

(vi) $(\sim p \wedge q) \wedge (\sim q \wedge (p \to q))$

Solution (i) Truth table for $(\sim q \wedge (p \to q)) \to \sim p$ is

p	q	$\sim p$	$\sim q$	$p \to q$	$\sim q \wedge (p \to q)$	$(\sim q \wedge (p \to q)) \to \sim p$
T	T	F	F	T	F	T
T	F	F	T	F	F	T
F	T	T	F	T	F	T
F	F	T	T	T	F	T

From the truth table we find that the compound proposition $(\sim q \wedge (p \to q)) \to$ ... assumes only the truth values T.

So, it is a tautology.

(ii) Truth table for $\sim p \to (p \to q)$ is

p	q	
T	T	
T	F	
F	T	
F	F	

From the truth table we find ... assumes only the truth value...

(iii) Truth table for $((p \to q) \wedge (q$ ...

p	q	r	$p \to q$	$p \to r$	$q \to r$
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	F	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	T	F	T	T	F
F	F	T	T	T	T
F	F	F	T	T	T

From the truth table we f...

$((p \to q) \wedge (q \to r)) \to (p$ ...

So, it is a tautology.

(iv) Truth table for $(p \vee q) \wedge$ ...

p	q	$\sim p$	
T	T	F	
T	F	F	
F	T	T	
F	F	T	

From the truth table w...

$(p \vee q) \wedge (\sim p \wedge \sim q)$ ...

So, it is a contradictio...

(v) Truth table for $\sim (q \to$ ...

p	q	r	$p \to q$	q...
T	T	T	T	
T	T	F	T	
T	F	T	F	
T	F	F	F	
F	T	T	T	
F	T	F	T	
F	F	T	T	
F	F	F	T	

From the truth tab...

$\sim (q \to r) \wedge r \wedge (p \to$ ...

(ii) Truth table for $\sim p \rightarrow (p \rightarrow q)$ is

p	q	$\sim p$	$p \rightarrow q$	$\sim p \rightarrow (p \rightarrow q)$
T	T	F	T	T
T	F	F	F	T
F	T	T	T	T
F	F	T	T	T

From the truth table we find that the compound proposition $\sim p \rightarrow (p \rightarrow q)$ assumes only the truth values T. So, it is a tautology.

(iii) Truth table for $((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$ is

p	q	r	$p \rightarrow q$	$p \rightarrow r$	$q \rightarrow r$	$(p \rightarrow q) \wedge (q \rightarrow r)$	$((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	T	F	T
T	F	F	F	F	T	F	T
F	T	T	T	T	T	T	T
F	T	F	T	T	F	F	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

From the truth table we find that the compound proposition $((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$ assumes only the truth values T.

So, it is a tautology.

(iv) Truth table for $(p \vee q) \wedge (\sim p \wedge \sim q)$ is

p	q	$\sim p$	$\sim q$	$p \vee q$	$(\sim p \wedge \sim q)$	$(p \vee q) \wedge (\sim p \wedge \sim q)$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	F

From the truth table we find that the compound proposition $(p \vee q) \wedge (\sim p \wedge \sim q)$ assumes only the truth values F.

So, it is a contradiction.

(v) Truth table for $\sim(q \rightarrow r) \wedge r \wedge (p \rightarrow q)$ is

p	q	r	$p \rightarrow q$	$q \rightarrow r$	$\sim(q \rightarrow r)$	$\sim(q \rightarrow r) \wedge r$	$\sim(q \rightarrow r) \wedge r \wedge (p \rightarrow q)$
T	T	T	T	T	F	F	F
T	T	F	T	F	T	F	F
T	F	T	F	T	F	F	F
T	F	F	F	T	F	F	F
F	T	T	T	T	F	F	F
F	T	F	T	F	T	F	F
F	F	T	T	T	F	F	F
F	F	F	T	T	F	F	F

From the truth table we find that the compound proposition

...mes only the truth values F. So, it is a contradiction.

(vi) Truth table for $(\sim p \wedge q) \wedge (\sim q \wedge (p \rightarrow q))$ is

p	q	$\sim p$	$\sim q$	$\sim p \wedge q$	$p \rightarrow q$	$\sim q \wedge (p \rightarrow q)$	$(\sim p \wedge q) \wedge (\sim q \wedge (p \rightarrow$
T	T	F	F	F	T	F	F
T	F	F	T	F	F	F	F
F	T	T	F	T	T	F	F
F	F	T	T	F	T	T	F

From the truth table we find that the compound propo... $(\sim p \wedge q) \wedge (\sim q \wedge (p \rightarrow q))$ assumes only the truth values F. So, it is a contradi...

2. Which one of the following well formed formula is not a tautolog...
(i) $p \vee \sim p$ (ii) $p \vee \sim\sim p$ (iii) $\sim(p \rightarrow p)$ (iv) $q \rightarrow p \vee q$ [CU, 2...

Solution (iii) $\sim(p \rightarrow p) \equiv \sim(\sim p \vee p) \equiv \sim$ (T) [Complement law]
$\equiv$ F is a contradiction

3. When a statement form is called contradiction? Show that $(p \wedge \sim p)$ ... contradiction. [KU, 2...

Solution A propositional Formula or statement formula P = $f(p, q, r, ...$ known to be a Contradiction if it is False for every truth assignment for p, q ... This means that in the truth table of the formula every entry in the last col... will be False (F). The truth table for $(p \wedge \sim p)$ is

p	$\sim p$	$(p \wedge \sim p)$
T	F	F
T	F	F
F	T	F
F	T	F

Hence, $(p \wedge \sim p)$ is a contradiction.

4. What is tautology? Show that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology [VU, 2...

Solution A propositional statement P = $f(p, q, r, ...)$ is known to be a tauto... if it is true for every truth assignment for $p, q, r, ...$ This means that in the ... table of the formula every entry in the last column will be True (T).

The truth table for $(p \wedge q) \rightarrow (p \vee q)$ is

p	q	$p \wedge q$	$p \vee q$	$(p \wedge q) \rightarrow (p \vee q)$
T	T	T	T	T
T	F	F	T	T
F	T	F	T	T
F	F	F	F	T

From the truth table we find that the compound proposition $(p \wedge q) \rightarrow (p$ assumes only the truth values T. So, it is a tautology.

5. For the statements p an...

Solution The truth table for p ...

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

From the truth table we fin... assumes only the truth values ...

6. Show that the follow... truth tables :
(i) $(p \rightarrow q) \rightarrow (p \wedge q)$

Solution (i) Truth table fo...

p	q
T	T
T	F
F	T
F	F

From the truth table we ... a contingency, since the tru... on the truth values of the v...

(ii) Truth table for $(p \wedge \sim$...

p	q	$\sim p$
T	T	F
T	F	F
F	T	T
F	F	T

From the truth table ... is a contingency

(iii) Truth table for $\sim(p$ ...

p	q	$\sim p$
T	T	F
T	F	F
F	T	T
F	F	T

From the truth tab...
$\sim (p \wedge q) \vee (\sim p \wedge \sim$...

5. **For the statements p and q verify that $p \to (q \to (p \wedge q))$ is a tautology** [BU, 2019]

Solution The truth table for $p \to (q \to (p \wedge q))$ is

p	q	$p \wedge q$	$q \to (p \wedge q)$	$p \to (q \to (p \wedge q))$
T	T	T	T	T
T	F	F	T	T
F	T	F	F	T
F	F	F	T	T

From the truth table we find that the compound proposition $p \to (q \to (p \wedge q))$ assumes only the truth values T. So, it is a tautology.

6. **Show that the following compound propositions are contingency using truth tables :**
(i) $(p \to q) \to (p \wedge q)$ (ii) $(p \wedge \sim q) \vee (\sim p \wedge q)$ (iii) $\sim (p \wedge q) \vee (\sim p \wedge \sim q)$

Solution (i) Truth table for $(p \to q) \to (p \wedge q)$ is

p	q	$p \to q$	$p \wedge q$	$(p \to q) \to (p \wedge q)$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	F
F	F	T	F	F

From the truth table we find that the compound proposition $(p \to q) \to (p \wedge q)$ is a contingency, since the truth table assumes true (T) and false (F) both depending on the truth values of the variables.

(ii) Truth table for $(p \wedge \sim q) \vee (\sim p \wedge q)$ is

p	q	$\sim p$	$\sim q$	$p \wedge \sim q$	$\sim p \wedge q$	$(p \wedge \sim q) \vee (\sim p \wedge q)$
T	T	F	F	F	F	F
T	F	F	T	T	F	T
F	T	T	F	F	T	T
F	F	T	T	F	F	F

From the truth table we find that the compound proposition $(p \wedge \sim q) \vee (\sim p \wedge q)$ is a contingency

(iii) Truth table for $\sim (p \wedge q) \vee (\sim p \wedge \sim q)$ is

p	q	$\sim p$	$\sim q$	$p \wedge q$	$\sim (p \wedge q)$	$\sim p \wedge \sim q$	$\sim (p \wedge q) \vee (\sim p \wedge \sim q)$
T	T	F	F	T	F	F	F
T	F	F	T	F	T	F	T
F	T	T	F	F	T	F	T
F	F	T	T	F	T	T	T

From the truth table we find that the compound proposition $\sim (p \wedge q) \vee (\sim p \wedge \sim q)$ is a contingency.

7. **Using truth table prove the equivalences**

(i) $p \to (q \vee r) \equiv (p \to q) \vee (p \to r)$

(ii) $p \wedge ((\sim q \vee r) \wedge (\sim r \vee q)) \equiv p \wedge (q \leftrightarrow r)$

(iii) $\sim (p \to q) \equiv p \wedge (\sim q)$

Solution **(i)** Truth table for $p \to (q \vee r)$ is

p	q	r	$q \vee r$	$p \to (q \vee r)$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	F	F
F	T	T	T	T
F	T	F	T	T
F	F	T	T	T
F	F	F	F	T

Truth table for $(p \to q) \vee (p \to r)$ is

p	q	r	$p \to q$	$p \to r$	$(p \to q) \vee (p \to r)$
T	T	T	T	T	T
T	T	F	T	F	T
T	F	T	F	T	T
T	F	F	F	F	F
F	T	T	T	T	T
F	T	F	T	T	T
F	F	T	T	T	T
F	F	F	T	T	T

Since the truth tables of $p \to (q \vee r)$ and $(p \to q) \vee (p \to r)$ are identical $\equiv$

$p \to (q \vee r) \; (p \to q) \vee (p \to r)$

(ii) Truth table for $p \wedge ((\sim q \vee r) \wedge (\sim r \vee q))$ is

p	q	r	$\sim q$	$\sim r$	$\sim q \vee r$	$\sim r \vee q$	$(\sim q \vee r) \wedge (\sim r \vee q)$	$p \wedge (\sim q \vee r) \wedge (\sim r$
T	T	T	F	F	T	T	T	T
T	T	F	F	T	F	T	F	F
T	F	T	T	F	T	F	F	F
T	F	F	T	T	T	T	T	T
F	T	T	F	F	T	T	T	F
F	T	F	F	T	F	T	F	F
F	F	T	T	F	T	F	F	F
F	F	F	T	T	T	T	T	F

Truth table for $p \wedge (q \leftrightarrow r)$ is

p	q	r	
T	T	T	
T	T	F	
T	F	T	
T	F	F	
F	T	T	
F	T	F	
F	F	T	
F	F	F	

Since the truth tables of $p \wedge ((\sim$

so $p \wedge ((\sim q \vee r) \wedge (\sim r \vee q)) \equiv p \wedge$

(iii) Truth table for $\sim (p \to q)$ is

p	q	
T	T	
T	F	
F	T	
F	F	

Truth table for is $p \wedge (\sim q)$

p	
T	
T	
F	
F	

Since the truth tables of $\sim$

$\sim (p \to q) \equiv p \wedge (\sim q)$

8. **$p \to (q \wedge \sim q)$ is logic**

(i) p (ii) $\sim p$ (iii) q (

Solution $p \to (q \wedge \sim q) \equiv p$

$\equiv \sim$

$\equiv$

Hence, option (ii) is cor

9. **Show that $p \vee (q \wedge$**

Solution The truth table

Truth table for $p \wedge (q \leftrightarrow r)$ is

p	q	r	$q \leftrightarrow r$	$p \wedge (q \leftrightarrow r)$
T	T	T	T	T
T	T	F	F	F
T	F	T	F	F
T	F	F	T	T
F	T	T	T	F
F	T	F	F	F
F	F	T	F	F
F	F	F	T	F

Since the truth tables of $p \wedge ((\sim q \vee r) \wedge (\sim r \vee q))$ and $p \wedge (q \leftrightarrow r)$ are identical so $p \wedge ((\sim q \vee r) \wedge (\sim r \vee q)) \equiv p \wedge (q \leftrightarrow r)$

(iii) Truth table for $\sim (p \rightarrow q)$ is

p	q	$p \rightarrow q$	$\sim (p \rightarrow q)$
T	T	T	F
T	F	F	T
F	T	T	F
F	F	T	F

Truth table for is $p \wedge (\sim q)$

p	q	$\sim q$	$p \wedge (\sim q)$
T	T	F	F
T	F	T	T
F	T	F	F
F	F	T	F

Since the truth tables of $\sim (p \rightarrow q)$ and $p \wedge (\sim q)$ are identical so

$\sim (p \rightarrow q) \equiv p \wedge (\sim q)$

8. **$p \rightarrow (q \wedge \sim q)$ is logically equivalent to** [CU, 2021]

(i) p (ii) $\sim p$ (iii) q (iv) $\sim q$

Solution $p \rightarrow (q \wedge \sim q) \equiv p \rightarrow F$ [Complement law]

$\equiv \sim p \vee F$

$\equiv \sim p$ [Identity law]

Hence, option (ii) is correct

9. **Show that $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ are logically equivalent** [VU, 2018]

Solution The truth table for $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ are

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$
T	T	T	T	T
T	T	F	F	T
T	F	T	F	T
T	F	F	F	T
F	T	T	T	T
F	T	F	F	F
F	F	T	F	F
F	F	F	F	F

and

p	q	r	$p \vee q$	$p \vee r$	$(p \vee q) \wedge (p \vee r)$
T	T	T	T	T	T
T	T	F	T	T	T
T	F	T	T	T	T
T	F	F	T	T	T
F	T	T	T	T	T
F	T	F	T	F	F
F	F	T	F	T	F
F	F	F	F	F	F

Since the truth tables of $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ are identical so $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

10. **Verify whether $(\sim(p \wedge q))$ is logically equivalent to $(\sim(p) \vee (\sim q))$ [KU**

Solution The truth table for $(\sim(p \wedge q))$ and $(\sim(p) \vee (\sim q))$ are

p	q	$p \wedge q$	$(\sim(p \wedge q)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

and

p	q	$\sim p$	$\sim q$	$(\sim(p) \vee (\sim q)$
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

Hence, $(\sim(p \wedge q))$ is logically equivalent to $(\sim(p) \vee (\sim q))$

11. **Suppose A and B are two statements forms of propositional logic.**

(i) When is A said to logically imply B?

(ii) When are A and B said to be logically equivalent

Show that $\sim p \rightarrow (q \rightarrow r)$ and $q \rightarrow (p \vee r)$ are logically equivalent.

[CU, 20

Solution Let A and B are two compo
A then B" is called conditional or implic
by $A \rightarrow B$ or $A \Rightarrow B$ (read as A implies B
or **premise** or **hypothesis** and the state
This is a compound proposition which
and true (T) otherwise.

Two compound propositions A a
simply equivalent, if they have ident
equal to the truth value of B for eve
assigned for the propositions p, q, r,
as A is equivalent to B).

The truth table for $\sim p \rightarrow (q \rightarrow r$

p	q	r
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

p	q
T	T
T	T
T	F
T	F
F	T
F	T
F	F
F	F

Since the truth tables of a

12. **Without using truth**

(i) $(\sim p \vee q) \wedge (p \wedge$

(ii) $p \rightarrow (q \rightarrow p) \equiv \sim$

(iii) $\sim(p \leftrightarrow q) \equiv (p \vee$

(iv) $p \rightarrow (q \vee r) \equiv (p$

Solution (i) L.H.S $= (\sim p \vee$

$\equiv (\sim p \vee q) \wedge ((p \wedge p) \wedge$

$\equiv (\sim p \vee q) \wedge (p \wedge q), b$

$\equiv (p \wedge q) \wedge (\sim p \vee q),$

Solution Let A and B are two compound propositions, then the proposition "if A then B" is called conditional or implication proposition of A and B. It is denoted by $A \to B$ or $A \Rightarrow B$ (read as A implies B). The statement A is called the **antecedent** or **premise** or **hypothesis** and the statement B is called **consequent** or **conclusion**. This is a compound proposition which is false when p is true (T) and q is false (F) and true (T) otherwise.

Two compound propositions A and B are said to be logically equivalent or simply equivalent, if they have identical truth values, viz. if the truth value of A is equal to the truth value of B for every one of the 2^n possible sets of truth values assigned for the propositions $p, q, r, \ldots$ This equivalence is denoted by $A \equiv B$ (read as A is equivalent to B).

The truth table for $\sim p \to (q \to r)$ and $q \to (p \wedge r)$ are

p	q	r	$\sim p$	$q \to r$	$\sim p \to (q \to r)$
T	T	T	F	T	T
T	T	F	F	F	T
T	F	T	F	T	T
T	F	F	F	T	T
F	T	T	T	T	T
F	T	F	T	F	F
F	F	T	T	T	T
F	F	F	T	T	T

and

p	q	r	$p \vee r$	$q \to (p \vee r)$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	T	T
F	T	T	T	T
F	T	F	F	F
F	F	T	T	T
F	F	F	F	T

Since the truth tables of and are identical so $\sim p \to (q \to r) \equiv q \to (p \vee r)$

12. **Without using truth tables, prove the following**

(i) $(\sim p \vee q) \wedge (p \wedge (p \wedge q)) \equiv p \wedge q$

(ii) $p \to (q \to p) \equiv \sim p \to (p \to q)$

(iii) $\sim(p \leftrightarrow q) \equiv (p \vee q) \wedge \sim (p \wedge q) \equiv (p \wedge \sim q) \vee (\sim p \wedge q)$

(iv) $p \to (q \vee r) \equiv (p \to q) \vee (p \to r)$

Solution (i) L.H.S = $(\sim p \vee q) \wedge (p \wedge (p \wedge q))$

$\equiv (\sim p \vee q) \wedge ((p \wedge p) \wedge q)$, by associative law

$\equiv (\sim p \vee q) \wedge (p \wedge q)$, by idempotent law

$\equiv (p \wedge q) \wedge (\sim p \vee q)$, by commutative law

$\equiv ((p \wedge q) \wedge \sim p) \vee ((p \wedge q) \wedge q)$, by distributive law

$\equiv ((\sim p \wedge (p \wedge q)) \vee ((p \wedge q) \wedge q))$, by commutative law

$\equiv ((\sim p \wedge p) \wedge q) \vee (p \wedge (q \wedge q))$, by associative law

$\equiv (F \wedge q) \vee (p \wedge q)$, by complement and idempotent law

$\equiv F \vee (p \wedge q)$, by dominant law

$\equiv p \wedge q$, by Identity law

Also R.H.S $= p \wedge q$

Hence, $(\sim p \vee q) \wedge (p \wedge (p \wedge q)) \equiv p \wedge q$

(ii) L.H.S $= p \to (q \to p)$

$\equiv \sim p \vee (q \to p)$ [as $p \to q \equiv \sim p \vee q$]

$\equiv \sim p \vee (\sim q \vee p)$ [as $p \to q \equiv \sim p \vee q$]]

$\equiv (\sim p \vee \sim q) \vee p$, by associative law

$\equiv (\sim q \vee \sim p) \vee p$, by commutative law

$\equiv \sim q \vee (\sim p \vee p)$, by associative law

$\equiv \sim q \vee (p \vee \sim p)$, by commutative law

$\equiv \sim q \vee T$, by complement law

$\equiv T$, by dominant law

Also, R.H.S $\equiv \sim p \to (p \to q)$

$\equiv p \vee (p \to q)$ [as $p \to q \equiv \sim p \vee q$]

$\equiv p \vee (\sim p \vee q)$ [as $p \to q \equiv \sim p \vee q$]

$\equiv (p \vee \sim p) \vee q$, by associative law

$\equiv T \vee q$, by complement law

$\equiv T$, by dominant law

Since, L.H.S = R.H.S so $p \to (q \to p) \equiv \sim p \to (p \to q)$

(iii) L.H.S $= \sim (p \leftrightarrow q)$

$\equiv \sim ((p \to q) \wedge (q \to p))$ [as $p \leftrightarrow q \equiv \sim((p \to q) \wedge (q \to p))$]

$\equiv \sim ((\sim p \vee q) \wedge (\sim q \vee p))$ [as $p \to q \equiv \sim p \vee q$]]

$\equiv \sim [((\sim p \vee q) \wedge \sim q) \vee ((\sim p \vee q) \wedge p)]$, by distributive law

$\equiv \sim [((\sim p \wedge \sim q) \vee (q \wedge \sim q)) \vee ((\sim p \wedge p) \vee (q \wedge p))]$, by distributive law

$\equiv \sim [((\sim p \wedge \sim q) \vee F)) \vee ((F \vee (q \wedge p))]$, by complement law

$\equiv \sim ((\sim p \wedge \sim q) \vee (q \wedge p))$, by identity law

$\equiv \sim (\sim (p \vee q)) \vee (q \wedge p))$, by De Morgan's law

$\equiv (p \vee q) \wedge \sim (q \wedge p)$, by De Morgan's law (1)

$\equiv (p \vee q) \wedge (\sim q \vee \sim p)$, by De Morgan's law

$\equiv ((p \vee q) \wedge \sim q) \vee ((p \vee q) \wedge \sim p)$, by distributive law

$\equiv ((p \wedge \sim q) \vee (q \wedge \sim q)) \vee ((p \wedge \sim p) \vee (q \wedge \sim p))$, by distributive law

$\equiv ((p \wedge \sim q) \vee F)) \vee ((F \vee (q \wedge \sim p))$, by complement law

$\equiv (p \wedge \sim q) \vee (q \wedge \sim p))$, by identity law

$\equiv (p \wedge \sim q) \vee (\sim p \wedge q))$, by commutative law

From (1) and (2), it follows th

$\sim(p \leftrightarrow q) \equiv (p \vee q) \wedge \sim(p \wedge$

(iv) R.H.S $= (p \to q) \vee (p \to r)$

$\equiv (\sim p \vee q) \vee (\sim p \vee r)$ [as p

$\equiv (q \vee \sim p) \vee (\sim p \vee r)$, by com

$\equiv q \vee (\sim p \vee (\sim p \vee r))$, by as

$\equiv q \vee ((\sim p \vee \sim p) \vee r))$, by a

$\equiv q \vee (\sim p \vee r))$, by idempot

$\equiv q \vee (r \vee \sim p))$, by commut

$\equiv (q \vee r) \vee \sim p$, by associati

$\equiv \sim p \vee (q \vee r)$, by commut

$\equiv p \to (q \vee r)$ [as $p \to q \equiv$

L.H.S $= p \to (q \vee r)$

Since, L.H.S = R.H.S so $p \to$

13. **Without using truth tabl**

(i) $(p \wedge q) \to (p \vee q)$ **is a**

(ii) $\sim (p \vee q) \vee (\sim p \wedge q) \vee$

(iii) $\sim((p \vee (\sim p \wedge q)) \vee (\sim$

Solution **(i)** $(p \wedge q) \to (p \vee q) \equiv$

$\equiv (\sim p \vee \sim q) \vee (p \vee q)$, by De

$\equiv (\sim p \vee \sim q) \vee p) \vee q$, by as

$\equiv (\sim p \vee (\sim q \vee p)) \vee q$, by a

$\equiv (\sim p \vee (p \vee \sim q)) \vee q$, by co

$\equiv ((\sim p \vee p) \vee \sim q) \vee q$, by as

$\equiv ((T) \vee \sim q) \vee q$, by comple

$\equiv (T \vee (\sim q \vee q))$, by associ

$\equiv T \vee T$, by complement la

$\equiv T$, by idempotent law

Hence, $(p \wedge q) \to (p \vee q)$ is a t

(ii) $\sim (p \vee q) \vee (\sim p \wedge q) \vee p \equiv ((\sim$

$\equiv (\sim p \wedge (\sim q \vee q)) \vee p$, by

$\equiv (\sim p \wedge T) \vee p$, by compl

$\equiv \sim p \vee p$, by identity law

$\equiv T$, by complement law

Hence, $\sim (p \vee q) \vee (\sim p \wedge q) \vee$

(iii) $\sim ((p \vee (\sim p \wedge q)) \vee (\sim q \wedge p))$

$\equiv \sim ((T \wedge q) \vee (\sim q \wedge p)) \wedge$

From (1) and (2), it follows that

$\sim(p \leftrightarrow q) \equiv (p \vee q) \wedge \sim(p \wedge q) \equiv (p \wedge \sim q) \vee (\sim p \wedge q)$

(iv) R.H.S $= (p \to q) \vee (p \to r)$

$\equiv (\sim p \vee q) \vee (\sim p \vee r)$ [as $p \to q \equiv \sim p \vee q$]

$\equiv (q \vee \sim p) \vee (\sim p \vee r)$, by commutative law

$\equiv q \vee (\sim p \vee (\sim p \vee r))$, by associative law

$\equiv q \vee ((\sim p \vee \sim p) \vee r))$, by associative law

$\equiv q \vee (\sim p \vee r))$, by idempotent law

$\equiv q \vee (r \vee \sim p))$, by commutative law

$\equiv (q \vee r) \vee \sim p$, by associative law

$\equiv \sim p \vee (q \vee r)$, by commutative law

$\equiv p \to (q \vee r)$ [as $p \to q \equiv \sim p \vee q$]

L.H.S $= p \to (q \vee r)$

Since, L.H.S = R.H.S so $p \to (q \vee r) \equiv (p \to q) \vee (p \to r)$

13. Without using truth tables, prove the following

(i) $(p \wedge q) \to (p \vee q)$ is a tautology

(ii) $\sim(p \vee q) \vee (\sim p \wedge q) \vee p$ is a tautology

(iii) $\sim((p \vee (\sim p \wedge q)) \vee (\sim q \wedge p)) \wedge q$ is a contradiction

Solution (i) $(p \wedge q) \to (p \vee q) \equiv \sim(p \wedge q) \vee (p \vee q)$ [as $p \to q \equiv \sim p \vee q$]

$\equiv (\sim p \vee \sim q) \vee (p \vee q)$, by De Morgan's law

$\equiv (\sim p \vee \sim q) \vee p) \vee q$, by associative law

$\equiv (\sim p \vee (\sim q \vee p)) \vee q$, by associative law

$\equiv (\sim p \vee (p \vee \sim q)) \vee q$, by commutative law

$\equiv ((\sim p \vee p) \vee \sim q) \vee q$, by associative law

$\equiv ((T) \vee \sim q) \vee q$, by complement law

$\equiv (T \vee (\sim q \vee q))$, by associative law

$\equiv T \vee T$, by complement law

$\equiv T$, by idempotent law

Hence, $(p \wedge q) \to (p \vee q)$ is a tautology

(ii) $\sim(p \vee q) \vee (\sim p \wedge q) \vee p \equiv ((\sim p \wedge \sim q) \vee (\sim p \wedge q)) \vee p$, by De Morgan's law

$\equiv (\sim p \wedge (\sim q \vee q)) \vee p$, by distributive law

$\equiv (\sim p \wedge T) \vee p$, by complement law

$\equiv \sim p \vee p$, by identity law

$\equiv T$, by complement law

Hence, $\sim(p \vee q) \vee (\sim p \wedge q) \vee p$ is a tautology

(iii) $\sim((p \vee (\sim p \wedge q)) \vee (\sim q \wedge p)) \wedge q \equiv \sim((p \vee \sim p) \wedge q)) \vee (\sim q \wedge p)) \wedge q$, by associative law

$\equiv \sim((T \wedge q)) \vee (\sim q \wedge p)) \wedge q$, by complement law

$\equiv \sim(T) \vee (\sim q \wedge p)) \wedge q$, by identity law

$\equiv \sim(T) \wedge (p \vee q)) \wedge q$, by idempotent and complement laws

$\equiv \sim(T \wedge (p \vee q)) \wedge q$, by dominant law

$\equiv \sim(p \vee q)) \wedge q$, by identity law

$\equiv (\sim p \wedge \sim q) \wedge q$, by De Morgan's law

$\equiv \sim p \wedge F$, by complement and associative law

$\equiv F$, by dominant law

Hence, $\sim((p \vee ((\sim p \wedge q)) \vee (\sim q \wedge p)) \wedge q$ is a contradiction

2.9 Axiomatic System of Propositional Logic

Truth tables play very important role to slve several problems concerni truth functional connectives that are to verify whether the statement fo is a tautology, or contradiction, or contingency, or whether two statemen logically equivalent. But there are several areas of complicated form of where the conclusion can not be generated and handled by truth tables. In cases we need another approach, the formal axiomatic theory to solve complicated logic. In mathematical logic, a formal theory is a set of sent written in a formal language.

2.10 Well-formed Formula

Mathematical logic, Propositional Calculus and Predicate logic us symbolic language to represent the logical structure of a statement, or *form* a compound proposition. Like any language, this symbolic language has r of *syntax*—grammatical rules for putting symbols together in the proper way in right order. An expression that obeys the syntactic rules of propositional l or calculus is called a ***well-formed formula***, or ***WFF***.

The following are the rules of *WFF*

1. A capital letter by itself is a *WFF*.
2. Any *WFF* can be prefixed with "~". This will also give a *WFF*
3. Any two *WFFs* can be put together with "•", "∨", "⊃", or "≡" between th enclosing the result in parentheses. This will also give a *WFF*
4. Parentheses in a statement formula may be omitted

Let T_j denote the formal theory in which there is a finite sequence of symb of T_j called *Expression*. Then there is a subset Σ of the set E of expressions o called the set of *WFF* of T_j

Example 2.11 : $p \wedge \sim q$ and $\sim((p \wedge \sim q) \supset \sim\sim r)$ are *WFF* whereas $p \wedge q$ $\sim((p \wedge \sim q) \supset \sim\sim r$ are not

2.11 Theory of Inferenc

Inference theory is concerned hypothesis or basic assumptions, c of reasoning, called rules of Infere of premises by using rules of infer **formal proof**. The rules of Inferen from a set of premises in a finite s are given in terms of statement statements or their truth values group of propositions called pre conclusion. Thus, the rule of Inf which receives premises, analyse is a finite set of relations $\{R_1, R_2$

This section deals with the r from a set of premises. Any con is called a valid conclusion an

Let, $P_1, P_2, \ldots, P_n$ is a g conclusion Q. Symbolically w

The argument is a valid a premises are true. The argum group of propositions implie $(P_1 \wedge P_2 \wedge \ldots \wedge P_n) \rightarrow Q$ is

The argument which is t

2.12 Rules of Inf

Rules of Inference are u derive a conclusion from a this process of derivation basic rules of inference c

Rule P	We may introdu
Rule T	We may introdu by one or more
Rule CP	If we can deriv then we can d

2.13 Elementa

There are several w one or more rules of i propositions p, q, r, c

2.11 Theory of Inference

Inference theory is concerned with inferring of a conclusion from certain hypothesis or basic assumptions, called premises, by applying certain principles of reasoning, called rules of Inference. When a conclusion is derived from a set of premises by using rules of inference, the process of such derivation is called a **formal proof**. The rules of Inference are only means used to draw a conclusion from a set of premises in a finite sequence of steps, called **argument**. These rules are given in terms of statement formulae rather than in terms of any specific statements or their truth values. Hence, an argument is an assertion; that a group of propositions called premises, yields another proposition, which is the conclusion. Thus, the rule of Inference is a logical form consisting of a function which receives premises, analyses their syntax and derives a conclusion. So, there is a finite set of relations $\{R_1, R_2, ..., R_n\}$ among *WFF* called rules of Inference.

This section deals with the rules of Inference by which conclusions are derived from a set of premises. Any conclusion which is arrived at by following these rules is called a valid conclusion and the argument is called a valid argument.

Let, $P_1, P_2, ..., P_n$ is a group of compound propositions that yields the conclusion Q. Symbolically we write as $P_1, P_2, ..., P_n \vdash Q$

The argument is a valid argument if the conclusion is true whenever all the premises are true. The argument is also valid if and only if the conjunctions of the group of propositions implies conclusion is a tautology i.e.,

$(P_1 \wedge P_2 \wedge ... \wedge P_n) \rightarrow Q$ is a tautology.

The argument which is not valid is called **fallacy** or invalid argument.

2.12 Rules of Inference

Rules of Inference are used to infer a conclusion from certain premises. If we derive a conclusion from a particular set of premises by using rules of inference, this process of derivation is called formal proof or deduction. Now, we state three basic rules of inference called rules of P, T and CP as follows :

Rule P	We may introduce a premise at any point of a derivation
Rule T	We may introduce a formula 'S' in the derivation if 'S' is tautologically implied by one or more preceding formulae in the derivation
Rule CP	If we can derive a formula 's' from another formula 'r' and a set of premises, then we can derive the statement $r \rightarrow s$ from the set of premises alone

2.13 Elementary Valid Argument Forms

There are several ways of providing a theorem. All of them are based on one or more rules of inference which are different forms of arguments. For the propositions p, q, r, certain rules of Inference are shown in the following table

Table 2.18 : Laws of Inference

Sl. No.	Rule of Inference	Rule in tautological form	Name of Rule
1	$p \wedge q$ $\therefore p$ $p \wedge q$ $\therefore q$	$(p \wedge q) \rightarrow p$ $(p \wedge q) \rightarrow q$	Simplification
2	p $\therefore p \vee q$ q $\therefore p \vee q$	$p \rightarrow (p \vee q)$ $q \rightarrow (p \vee q)$	Addition
3	p q $\therefore p \wedge q$	$((p) \wedge ((q)) \rightarrow (p \wedge q)$	Conjunction
4	p $p \rightarrow q$ $\therefore q$	$(p \wedge (p \rightarrow q)) \rightarrow q$	Modus ponens (Law of detachment)
5	$\neg q$ $p \rightarrow q$ $\therefore \neg p$	$((\neg q) \wedge (p \rightarrow q)) \rightarrow \neg p$	Modus tollens (Law of Contra positive)
6	$p \rightarrow q$ $q \rightarrow r$ $\therefore p \rightarrow r$	$((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$	Hypothetical Syllogism (Chain rule)
7	$p \vee q$ $\neg p$ $\therefore q$	$((p \vee q) \wedge \neg p) \rightarrow q$	Disjunctive Syllogism
8	$p \vee q$ $\neg p \vee r$ $\therefore q \vee r$	$((p \vee q) \wedge (\neg p \vee r)) \rightarrow (q \vee r)$	Resolution
9	$p \vee q$ $p \rightarrow r$ $q \rightarrow r$ $\therefore r$	$((p \vee q) \wedge (p \rightarrow r)) \wedge (q \rightarrow r)) \rightarrow r$	Dilemma

In the following Section, we shall discuss some of the rules given in the table

2.13.1 Law of detachment (Modus ponens)

Modus ponens is a Latin term which means 'Method of Affirmation'. Modus ponens states that if both an implication and its hypothesis are true, then the conclusion of this implication is true.

Using truth table we can show the validity of the rule Modus ponens. The truth table of the rule Modus ponens is

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Since, $((p \rightarrow q) \wedge p) \rightarrow q$ is a ta

Illustration : Suppose that get wet in rain" and the hyp ponens, it follows that the c rain", is true. This is illustrat Let us consider the followin

p : It rains today

q : We will get wet in rain

Then the premises $p \rightarrow q$ ponens

Example 2.12 : Show that "If I study hard, then I wi I work hard. Therefore, I stood first i

Solution : Let us consid

p : It study hard

q : I will stand first in th

In symbolic form

$p \rightarrow q$

p

$\therefore q$

Hence, $((p \rightarrow q) \wedge p)$ ponens".

2.13.2 Law of Co

Modus tollens means validity of the rule Modu

p	q	$\neg p$
T	T	F
T	F	F
F	T	T
F	F	T

Since, $((p \rightarrow q) \wedge \neg q)$

Table 2.19

p	q	$p \to q$	$(p \to q) \wedge p$	$((p \to q) \wedge p) \to q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Since, $((p \to q) \wedge p) \to q$ is a tautology, so the rule Modus ponens is valid

Illustration : Suppose that the implication "If it rains today, then we will get wet in rain" and the hypothesis "It is raining" are true. Then by Modus ponens, it follows that the conclusion of the implication, "We will get wet in rain", is true. This is illustrated as follows:

Let us consider the following simple propositions

p : It rains today

q : We will get wet in rain

Then the premises $p \to q$ and p shows $(p \wedge (p \to q)) \to q$ is valid by Modus ponens

Example 2.12 : Show that the following argument is valid by Modus ponens

"If I study hard, then I will stand first in the Examination.

I work hard.

Therefore, I stood first in the examination"

Solution : Let us consider the following simple propositions

p : It study hard

q : I will stand first in the Examination

In symbolic form

$p \to q$

p

$\therefore q$

Hence, $((p \to q) \wedge p) \to q$ is a tautology. This follows from the rule "Modus ponens".

2.13.2 Law of Contra position (Modus tollens)

Modus tollens means "Method of Denial". Using truth table we can show the validity of the rule Modus tollens. The truth table of the rule Modus tollens is

Table 2.20

p	q	$\sim p$	$\sim q$	$p \to q$	$(p \to q) \wedge \sim q$	$((p \to q) \wedge (\sim q)) \to \sim p$
T	T	F	F	T	F	T
T	F	F	T	F	F	T
F	T	T	F	T	F	T
F	F	T	T	T	T	T

Since, $((p \to q) \wedge \sim q) \to \sim p$ is a tautology, so the rule Modus tollens is valid

Illustration : Suppose that the implication "If it rains today, then ... wet in rain" and the hypothesis "we will not get wet in rain" are ... Modus tollens, it follows that the conclusion of the implication, ... rain today", is true. This is illustrated as follows:

Let us consider the following simple propositions

p : It rains today

q : We will get wet in rain

Then the premises $p \to q$ and $\neg q$ shows $((p \to q) \wedge \neg q) \to \neg p$ is valid ... tollens.

Example 2.13 : Show that the following argument is valid by Modus tollens.

"If I study hard, then I will stand first in the Examination.

I will not stand first in the Examination.

Therefore, I do not study hard"

Solution : Let us consider the following simple propositions

p : It study hard

q : I will stand first in the Examination

In symbolic form

$p \to q$

$\neg q$

$\therefore \neg p$

Hence, $((p \to q) \wedge \neg q) \to \neg p$ is a tautology. This follows from the rule "... tollens".

2.13.3 Hypothetical Syllogism

Hypothetical Syllogism is chain rule of implications. The validity ... argument depends on its forms but not on its contents. Using truth table ... show the validity of the rule Hypothetical Syllogism. The truth table of ... Hypothetical Syllogism is

Table 2.21

p	q	r	$p \to q$	$q \to r$	$p \to r$	$(p \to q) \wedge (q \to r)$	$((p \to q) \wedge (q \to r)) \to$ ...
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	T	F	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	T	F	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

Since, $((p \to q) \wedge (q \to r)) \to (p \to r)$ is a tautology, so the rule Hypothe... Syllogism is valid

Example 2.14 : State which rule of inference is used in the argument "If it rains today then we will not go to school today. If we do not go to school today, then we will go to school tomorrow. Therefore, if it rains today then we will go to school tomorrow".

Solution : Let us consider the following simple propositions

p : It rains today

q : we will not go to school today

r : we will go to school tomorrow

In symbolic form

$p \to q$

$q \to r$

$\therefore p \to r$

Hence, $((p \to q) \wedge (q \to r)) \to p \to r$ is a tautology. This follows from the rule "Hypothetical Syllogism".

Example 2.15 : State which rule of inference is used in the argument "If I do my homework, then my teacher will be happy

If my teacher will be happy, then I will eat chicken

Therefore, if I do my homework, then I will eat chicken"

Solution : Let us consider the following simple propositions

p : I do my homework

q : my teacher will be happy

r : I will eat chicken

In symbolic form

$p \to q$

$q \to r$

$\therefore p \to r$

Hence, $((p \to q) \wedge (q \to r)) \to p \to r$ is a tautology. This follows from the rule "Hypothetical Syllogism".

2.13.4 Disjunctive Syllogism

Using truth table we can show the validity of the rule Disjunctive Syllogism. The truth table of the rule Disjunctive Syllogism is

Table 2.22

p	q	$\sim p$	$p \vee q$	$(p \vee q) \wedge \sim p$	$((p \vee q) \wedge \sim p) \to q$
T	T	F	T	F	T
T	F	F	T	F	T
F	T	T	T	T	T
F	F	T	F	F	T

Since, $((p \vee q) \wedge \sim p) \to q$ is a tautology, so the rule Disjunctive Syllogism is valid

Example 2.16 : State which rule of inference is used in the argument
"Either Rita is a good girl or Riya is a good girl.
Rita is not a good girl
Therefore, Riya is a good girl"

Solution : Let us consider the following simple propositions

p : Rita is a good girl
q : Riya is a good girl
In symbolic form
$p \vee q$
$\neg p$
$\therefore q$
Hence, $((p \vee q) \wedge \neg p) \rightarrow q$ is a tautology. This follows from the rule "Disjunc
Syllogism".

Example 2.17 : State which rule of inference is used in the argument
"It is very hot today. Therefore, it is either very hot today or it is raining"

Solution : Let us consider the following simple propositions

p : It is very hot today
q : It is raining
In symbolic form $p \rightarrow p \vee q$ is a tautology. Here, the rule of Inferen
"Addition".

Example 2.18 : Express the following argument symbolically and test whe
this is a valid argument using truth table :
"If I study, then I will pass examination.
If I do not go to picnic, then I will study.
But I failed examination.
Therefore, I went to picnic".

Solution : Let us consider the following simple propositions

p : I study
q : I will pass examination
r : I go to picnic
The first premise is $P_1 : p \rightarrow q$
The second premise is $P_2 : \neg r \rightarrow p$
The third premise is $P_3 : \neg q$
The conclusion is $Q : r$
Therefore, the argument is symbolically written as P_1, P_2, P_3 Q that is
$p \rightarrow q, \neg r \rightarrow p, \neg q \vdash Q$
This argument will be valid if $(P_1 \wedge P_2 \wedge P_3) \rightarrow Q$ that is
$((p \rightarrow q) \wedge (\neg r \rightarrow p) \wedge (\neg q)) \rightarrow r$ is a tautology

The truth table for $((p \rightarrow q) \wedge (\neg$

p	q	r	$\neg q$	$p \rightarrow q$	$\neg r \rightarrow$
T	T	T	F	T	T
T	T	F	F	T	T
T	F	T	T	F	T
T	F	F	T	F	T
F	T	T	F	T	T
F	T	F	F	T	F
F	F	T	T	T	T
F	F	F	T	T	F

We see all the truth values of
a valid argument

Example 2.19 : Express th
whether this is a valid argu
"If Rita solved this problem
Rita scores 100
Therefore, Rita solved this p

Solution : Let us consider
p : Rita solved this problem
q : She (Rita) scores 100
The first premise is $P_1 : p \rightarrow$
The second premise $P_2 : q$
The conclusion is $Q : p$
Therefore, the argument is
This argument will be valid
The truth table of $((p \rightarrow q)$

p	q	$p \rightarrow$
T	T	T
T	F	F
F	T	T
F	F	T

We see all the truth values
argument (fallacy).

2.14 Proof of valid

We can test the validity of
number of variable statement
efficient method to test the v

The truth table for $((p \to q) \wedge (\neg r \to p) \wedge (\neg q)) \to r$ is

Table 2.23

p	q	r	$\neg q$	$p \to q$	$\neg r \to p$	$(p \to q) \wedge (\neg r \to p) \wedge \neg q$	$(p \to q) \wedge (\neg r \to p) \wedge \neg q) \to r$
T	T	T	F	T	T	F	T
T	T	F	F	T	T	F	T
T	F	T	T	F	T	F	T
T	F	F	T	F	T	F	T
F	T	T	F	T	T	F	T
F	T	F	F	T	F	F	T
F	F	T	T	T	T	F	T
F	F	F	T	F	F	F	T

We see all the truth values of $((p \to q) \wedge (\neg r \to p) \wedge (\neg q)) \to r$ are "T". So, this is valid argument

Example 2.19 : Express the following argument symbolically and test whether this is a valid argument using truth table:
"If Rita solved this problem, then he scores 100
Rita scores 100
Therefore, Rita solved this problem"

Solution : Let us consider the following simple propositions

p: Rita solved this problem

q: She (Rita) scores 100

The first premise is $P_1 : p \to q$

The second premise $P_2 : q$

The conclusion is $Q : p$

Therefore, the argument is symbolically written as $P_1, P_2 \vdash Q$ i.e., $p \to q, q \vdash p$

This argument will be valid if $P_1 \wedge P_2 \to Q$ that is $((p \to q) \wedge q) \to p$ is a tautology

The truth table of $((p \to q) \wedge q) \to p$ is

Table 2.24

p	q	$p \to q$	$(p \to q) \wedge q$	$((p \to q) \wedge q) \to p$
T	T	T	T	T
T	F	F	F	T
F	T	T	T	F
F	F	T	F	T

We see all the truth values of $((p \to q) \wedge q) \to p$ are not "T". So, this is invalid argument (fallacy).

2.14 Proof of validity using Rules of Inference

We can test the validity of an argument constructing truth tables. But as the number of variable statements increases, the truth table grows unwieldy. A more efficient method to test the validity of the argument is to reduce its conclusion

from its premises by a sequence of elementary arguments each of which is k... to be valid. The following examples are given to describe the process of ... Rules of Inference.

Example 2.20 : Prove that the argument $p \to \neg q, r \to q, r \vdash \neg p$ is valid wi... using truth table.

Solution : The three premises are : $p \to \neg q$, $r \to q$ and r

The conclusion is : $\neg p$

Now, the following sequence of elementary arguments is used to pro... validity of the given argument:

(i) $p \to \neg q$	P ? (given, first premise)
(ii) $r \to q$	Rule P (given, second premise)
(iii) $\neg q \to \neg r$	Rule T (contrapositive of (ii))
(iv) $p \to \neg r$	Rule T (Hypothetical syllogism using (i) and ...
(v) $r \to \neg p$	Rule T (contapositive of (iv))
(vi) r is true	Rule P (given, third premise)
(vii) $\neg p$	Rule T (modus ponens, using (v) and (vi))

Hence, the given argument is valid

Example 2.21 : Prove that the argument $p \to (q \to s), \neg r \vee p, q \vdash r \to s$ is ... without using truth table.

Solution : The three premises are: $p \to (q \to s)$, $\neg r \vee p$ and q

The conclusion is : $r \to s$

Now, the following sequence of elementary arguments is used to pro... validity of the given argument:

(i) $\neg r \vee p$	Rule P (given, second premise)
(ii) r	Rule P (assumed premise)
(iii) p	Rule T (disjunctive syllogism using (i) and (ii))
(iv) $p \to (q \to s)$	Rule P (given, first premise)
(v) $q \to s$	Rule T (modus ponens, using (iii) and (iv))
(vi) q is true	Rule P (given, third premise)
(vii) s	Rule T (modus ponens, using (v) and (vi))
(viii) $r \to s$	Rule CP

Hence, the given argument is valid.

Example 2.22 : Prove that the argument $p \to q$, $q \to r$, $r \to s$, $\neg s$, $p \vee t$ ... valid without using truth tables.

Solution : The five premises are : $p \to q$, $q \to r$, $r \to s$, $\neg s$ and $p \vee t$

The conclusion is : t

Now, the following sequence of elementary arguments is used to pro... validity of the given argument :

(i) $p \to q$	Rule P (given, first premise)

Pro...

(i) ...	Rule P (given, second ...
(ii) ... $\to s$	Rule P (given, third p...
(iv) $\neg s$	Rule P (given, fourth ...
(v) $p \vee t$	Rule P (given, fifth p...
(vi) $p \to r$	Rule T (Hypothetical ...
(vii) $p \to s$	Rule T (Hypothetical ...
(viii) $\neg p$	Rule T (Modus tolle...
(ix) t	Rule T (Disjunctive ...

Hence, the given argument is valid.

2.15 Different Methods of Pro...

Different forms of arguments can be pu... statement in Logic. They are:

(i) Direct proof
(ii) Indirect proof
(iii) Proof by Cases
(iv) Mathematical Induction

Let us discuss about the different metho...

2.15.1 Direct Proof

A direct proof is based on Modus ponen... logical valid argument in which we start wi... "P" is true. Here, it is assumed that the stat... using $P(x_1, x_2, \ldots, x_n)$ as well as other law... derived theorems it can be proved directly... this let us consider the following example...

Example 2.23 : Give a direct proof of t... integers is even".

Solution : Let p and q be two simp... integers, then

$x = 2n$, $n \in \mathbb{Z}$ (set of integers)

$y = 2m$, $m \in \mathbb{Z}$

p : the product of two even integers ...

q : xy is even

Now, $xy = (2n)(2m) = 4mn = 2(2mn...$

$= 2t$, $t \in \mathbb{Z}$, which is an even ...

Therefore, p is true and $p \to q$ is tru... "Modus ponens".

Hence, the result follows.

(i) Rule P (given, second premise)
(ii) Rule P (given, third premise)
(iv) Rule P (given, fourth premise)
Rule P (given, fifth premise)
Rule T (Hypothetical syllogism using (i) and (ii))
Rule T (Hypothetical syllogism using (vi) and (iii))
Rule T (Modus tollens using (vii) and (iv)
Rule T (Disjunctive syllogism, using (v) and (viii))

Hence, the given argument is valid.

2.15 Different Methods of Proof

Different forms of arguments can be put together to prove or disprove a statement in Logic. They are:

(i) Direct proof
(ii) Indirect proof
(iii) Proof by Cases
(iv) Mathematical Induction

Let us discuss about the different methods of proofs of statements in logic.

2.15.1 Direct Proof

A direct proof is based on Modus ponens entirely. A direct proof of $P \rightarrow Q$ is a logical valid argument in which we start with the assumption that the hypothesis "P" is true. Here, it is assumed that the statement P $(x_1, x_2, ..., x_n)$ is true and then using $P(x_1, x_2, ..., x_n)$ as well as other laws, axioms, definitions and previously derived theorems it can be proved directly that $Q(x_1, x_2, ..., x_n)$ is true. To illustrate this let us consider the following examples.

Example 2.23 : Give a direct proof of the statement "The product of two even integers is even".

Solution : Let p and q be two simple propositions. Let x and y be two even integers, then

$x = 2n, n \in \mathbb{Z}$ (set of integers)

$y = 2m, m \in \mathbb{Z}$

p : the product of two even integers 'xy'

q : xy is even

Now, $xy = (2n)\ (2m) = 4mn = 2\ (2mn)$

$= 2t, t \in \mathbb{Z}$, which is an even integer.

Therefore, p is true and $p \rightarrow q$ is true, we conclude that q is true by the rule of "Modus ponens".

Hence, the result follows.

Example 2.24 : If $x \in \mathbb{R}$ (set of real numbers) such that $x^2 = k$ ...
$x = \sqrt{k}$ $\sqrt{k}$ or $x = -\sqrt{k}$

Solution : Let p and q be two simple propositions.

$p : x \in \mathbb{R}$ such that $x^2 = k$

$q : x = \sqrt{k}$ or $x = -\sqrt{k}$

Now, $x^2 = k \Rightarrow x = \pm\sqrt{k}$

Therefore, p is true and $p \to q$ is true, we conclude that q is true by "Modus ponens".

Hence, the result follows

2.15.2 Indirect Proof

An indirect proof of $P \to Q$ involves two different methods

a. Proof by Contrapositive
b. Proof by Contradiction

Let us discuss these methods with examples

a. Proof by Contrapositive

The method of contrapositive states that the implication $p \to q$ ... equivalent to its contrapositive $\sim q \to \sim p$

The truth table of $p \to q$ is

Table 2.25

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

The truth table of $\sim q \to \sim p$ is

Table 2.26

p	q	$\sim p \to \sim q$
T	T	T
T	F	F
F	T	T
F	F	T

Thus, $p \to q \equiv \sim q \to \sim p$

So, in order to prove that the implication $p \to q$ holds we assume ... false and then show that p is false. To illustrate this let us consider the ... example

Example 2.25 : Prove of the statement "If x is an integer and x^2 is ... x is also even", by proving its Contrapositive statement.

Solution : Let p and q be two simple propositions.

p : x is an integer and x^2 is even

q : x is an even integer

We need to prove $p \to q$

By using contrapositive $\sim q \to \sim p$

Now, $\sim q$ says x is odd integer. Thus $x = 2n$

Taking the square of both sides, we get,

$x^2 = (2n + 1)^2 = 4n^2 + 4n + 1 = (4n^2 + 4n)$

$= 2(2n^2 + 2n) + 1$

$= 2t + 1$; $t = (2n^2 + 2n) \in \mathbb{Z}$

$\Rightarrow x^2$ is odd integer.

Therefore, '$\sim p$' holds. Hence, the result ...

b. Proof by Contradiction

Proof by method of Contradiction is a ... is reduced to an absurdity. For the arg... is true then q is true. By the method of ... that $\sim q$ is true (q is false). Then by le... situation which will demand that our ... a statement that is always false. The... consider the following example

Example 2.26 : Show $\sqrt{3}$ that is irra...

Solution : Let p and q be two simpl...

p : x and y are positive integers

q : $\sqrt{3}$ is irrational

Let us consider $\sqrt{3}$ is rational ($\sim q$ is ...

Then, $\frac{x}{y} = \sqrt{3}$, x and y are relatively ...

This implies $x = \sqrt{3}\, y \Rightarrow x^2 = 3y^2$

$\Rightarrow 3 \mid x^2$ (3 divides x^2)

$\Rightarrow 3 \mid x$ (3 divides x)

Therefore, by definition, $x = 3t$ for ...

$x^2 = 9t^2$

But, $x^2 = 3y^2$ also

So, $3y^2 = 9t^2 \Rightarrow 3t^2 = y^2$

$\Rightarrow 3 \mid y^2$ (3 divides y^2)

$\Rightarrow 3 \mid y$ (3 divides y)

Equations (i) and (ii) implies that ... assumption that x and y are relati... rational is wrong. So, $\sqrt{3}$ is irrati...

p : x is an integer and x^2 is even

q : x is an even integer

We need to prove $p \rightarrow q$

By using contrapositive $\sim q \rightarrow \sim p$

Now, $\sim q$ says x is odd integer. Thus $x = 2n + 1$, $n \in \mathbb{Z}$

Taking the square of both sides, we get,

$x^2 = (2n + 1)^2 = 4n^2 + 4n + 1 = (4n^2 + 4n) + 1$

$= 2(2n^2 + 2n) + 1$

$= 2t + 1$; $t = (2n^2 + 2n) \in \mathbb{Z}$

$\Rightarrow x^2$ is odd integer.

Therefore, '$\sim p$' holds. Hence, the result is true by the method of contrapositive.

b. Proof by Contradiction

Proof by method of Contradiction is a method in which the given assumption is reduced to an absurdity. For the argument $p \rightarrow q$ we need to prove that if p is true then q is true. By the method of Contradiction we have to first consider that $\sim q$ is true (q is false). Then by logical arguments we need to arrive at a situation which will demand that our assumption ($\sim q$ is true) is a contradiction, a statement that is always false. Therefore, q is true. To illustrate this let us consider the following example

Example 2.26 : Show $\sqrt{3}$ that is irrational by the method of contradiction.

Solution : Let p and q be two simple propositions.

p : x and y are positive integers

q: $\sqrt{3}$ is irrational

Let us consider $\sqrt{3}$ is rational ($\sim q$ is true).

Then, $\frac{x}{y} = \sqrt{3}$, x and y are relatively prime.

This implies $x = \sqrt{3}\, y \Rightarrow x^2 = 3y^2$

$\Rightarrow 3 \mid x^2$ (3 divides x^2) ... (i)

$\Rightarrow 3 \mid x$ (3 divides x)

Therefore, by definition, $x = 3t$ for some $t \in \mathbb{Z}$

$x^2 = 9t^2$

But, $x^2 = 3y^2$ also

So, $3y^2 = 9t^2 \Rightarrow 3t^2 = y^2$

$\Rightarrow 3 \mid y^2$ (3 divides y^2) (ii)

$\Rightarrow 3 \mid y$ (3 divides y)

Equations (i) and (ii) implies that 3 divides both x and y which contradicts our assumption that x and y are relatively prime. Thus, our assumption that $\sqrt{3}$ is rational is wrong. So, $\sqrt{3}$ is irrational.

2.15.3 Proof by Cases

For the hypothesis $p_1, p_2, \ldots, p_n$ and conclusion q, the conditional st... $p_1 \vee p_2 \vee \ldots \vee p_n \rightarrow q \equiv (p_1 \rightarrow q) \wedge (p_2 \rightarrow q) \wedge \ldots \wedge (p_n \rightarrow q)$ holds. This ... process of proofs by considering different cases and finally combining ... come to the conclusion.

Example 2.27 : Prove that for every positive integer n, $n^2 + n$ is alwa... integer.

Solution : Let $p_1 : n$ is positive even integer

Let $p_2 : n$ is positive odd integer

$q : n^2 + n$ is positive even integer

◆ **Case-I :** To show $p_1 \rightarrow q$ is true

Let p_1 is true

Then n is a positive even integer

Let us consider $n = 2t$, where t is an integer

Now, $n^2 + n = 4t^2 + 2t = 2(2t^2 + t)$, which is always even integer

Thus, $p_1 \rightarrow q$ is true

◆ **Case-II :** To show $p_2 \rightarrow q$ is true

Let p_2 is true

Then n is a positive odd integer

Let us consider $n = 2t + 1$, where t is an integer

Now, $n^2 + n = (2t+1)^2 + (2t+1) = (4t^2 + 4t + 1) + (2t + 1)$

$= 4t^2 + 6t + 2 = 2(2t^2 + 3t + 1)$, which is always even integer

Thus, $p_2 \rightarrow q$ is true

So, from Case I and Case II, it follows that

$(p_1 \rightarrow q) \wedge (p_2 \rightarrow q)$ is true and hence

$p_1 \vee p_2 \rightarrow q$ is true since,

$p_1 \vee p_2 \rightarrow q \equiv (p_1 \rightarrow q) \wedge (p_2 \rightarrow q)$ holds

2.15.4 Mathematical Induction

Mathematical Induction is a technique for proving propositional sta... over positive integers. It is used extensively to prove results about a large ... of discrete objects. We state here two different statements of Mathe... Induction.

■ **Statement-I : First Principle of Induction :** Suppose there is a given sta... $P(n)$ involving the natural number n such that we have the follow... steps as

(i) Basic Step : The statement is true for $n = 1$, i.e., $P(1)$ is true, and

(ii) Inductive Step : If the statement is true for $n = k$ (where k is some ... integer), then the statement is also true for $n = k + 1$, i.e., truth ... implies the truth of $P(k + 1)$.

Then, $P(n)$ is true for all natural numbers n.

Example 2.28 : For all $n \geq 1$, prove tha...

Solution : Let the given statement be ...

i.e., $P(n) : 1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2...}{6}$

● **Basic Step :** For $n = 1$, $n(n+1)(2n+...$ which is true.

● **Inductive Step :** Now, assume that ...

i.e., $P(k) : 1^2 + 2^2 + \ldots + k^2 = \frac{k(k+1)}{...}$

We shall now prove that $P(k+1)$ i...

Now, we have

$1 + 2^2 + \ldots + k^2 + (k+1)^2 = \frac{k(k+1)}{...}$

$= \frac{k(k+1)(2k+1)+6(k+1)^2}{6} = ...$

Thus $P(k+1)$ is true, whenever ...

Hence, from the principle of ma... true for all natural numbers n.

■ **Statement-II : Second Principl...** statement $P(n)$ involving the natu... two steps as

(i) Basic Step : The statement is ...

(ii) Inductive Step : $P(1)$, $P(2)$, ... where k is some positive inte...

Example 2.29 : Prove that $(3 + ...$

Solution : Let the given statem...

$P(n) : (3 + \sqrt{7})^n + (3 - \sqrt{7})^n$ is ...

● **Basic Step :** For $n = 1$, $(3 + \sqrt{7}...$ which is true.

● **Inductive Step :** Now, assume ... positive integer k, i.e.,

$P(k) : (3 + \sqrt{7})^k + (3 - \sqrt{7})^k$ is ...

Now, $m^{k+1} + n^{k+1}$, where $m = ...$

$= (m + n)(m^k + n^k) - mn(...$

$= 6(m^k + n^k) - 2(m^{k-1} + n...$

This is an even integer, sinc...

This proves that $P(k+1)$ is t... positive integer k.

Hence, from the principle o... true for all natural number...

Example 2.28 : For all $n \geq 1$, prove that $1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$

Solution : Let the given statement be $P(n)$,

i.e., $P(n) : 1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$

- **Basic Step :** For $n = 1$, $n(n+1)(2n+1) = \frac{1(1+1)(2+1)}{6} = \frac{1(2)(3)}{6} = 1$ which is true.
- **Inductive Step :** Now, assume that $P(k)$ is true for some positive integer k,

i.e., $P(k) : 1^2 + 2^2 + \ldots + k^2 = \frac{k(k+1)(2k+1)}{6}$ ...(i)

We shall now prove that $P(k+1)$ is also true.

Now, we have

$$1 + 2^2 + \ldots + k^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \text{ [from (i)]}$$

$$= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} = \frac{(k+1)(2k^2+7k+6)}{6} = \frac{(k+1)(k+1+1)[2(k+1)+1)}{6}$$

Thus $P(k+1)$ is true, whenever $P(k)$ is true.

Hence, from the principle of mathematical induction, the statement $P(n)$ is true for all natural numbers n.

- **Statement-II : Second Principle of Induction :** Suppose there is a given statement $P(n)$ involving the natural number n such that we have the following two steps as

(i) **Basic Step :** The statement is true for $n = 1$, i.e., $P(1)$ is true, and

(ii) **Inductive Step :** $P(1)$, $P(2)$, ..., $P(k)$ is true implies $P(k+1)$ is also true, where k is some positive integer

Example 2.29 : Prove that $(3 + \sqrt{7})^n + (3 - \sqrt{7})^n$ is an even integer, $\forall n \in N$

Solution : Let the given statement be $P(n)$, i.e.,

$P(n) : (3 + \sqrt{7})^n + (3 - \sqrt{7})^n$ is an even integer

- **Basic Step :** For $n = 1$, $(3 + \sqrt{7}) + (3 - \sqrt{7}) = 6$ which is an even integer which is true.
- **Inductive Step :** Now, assume that $P(1)$, $P(2)$, $P(3)$, ..., $P(k)$ are true for some positive integer k, i.e.,

$P(k) : (3 + \sqrt{7})^k + (3 - \sqrt{7})^k$ is an even integer

Now, $m^{k+1} + n^{k+1}$, where $m = (3 + \sqrt{7})$ and $n = (3 - \sqrt{7})$

$= (m+n)(m^k + n^k) - mn(m^{k-1} + n^{k-1})$

$= 6(m^k + n^k) - 2(m^{k-1} + n^{k-1})$ [$\because m + n = 6$ and $mn = 2$]

This is an even integer, since $(m^k + n^k)$ and $(m^{k-1} + n^{k-1})$ are even integers

This proves that $P(k+1)$ is true whenever $P(1)$, $P(2)$, ..., $P(k)$ is true for some positive integer k.

Hence, from the principle of mathematical induction, the statement $P(n)$ is true for all natural numbers n.

Illustrative Examples

1. State which rule of inference is used in the argument

(i) It is below freezing now. Therefore, it is either below fre... raining now.

(ii) If today is Sunday then we will not go to office. If we do no... office today, then we will go to a long drive. Therefore, if to... Sunday then we will go for a long drive.

(iii) Either Heera is a good boy or Sanjay is a good boy. Heera i... good boy, therefore, Sanjay is good boy.

(iv) If a polygon is a square then it has four sides. If a polygon ha... sides, then it has three angles. Therefore, if a polygon is a s... then it has three angles.

(v) Bula is an excellent swimmer. If Bula is an excellent swimmer... she can work as a lifeguard. Therefore, Bula can work as a lifeg...

(vi) If I run fast, then I will be champion in the race. I do not be... champion in the race. Therefore, I do not run fast.

Solution (i) Let us consider the following simple propositions

p : It is below freezing now

q : It is raining now.

Therefore, $p \to p \vee q$

This follows from the rule "Addition".

(ii) Let us consider the following simple propositions

p : today is Sunday

q : we will not go to office

r: we will go to a long drive

In symbolic form

$p \to q$

$q \to r$

$\therefore p \to r$

Hence, $(p \to q) \wedge (q \to r) \to p \to r$ is a tautology. This follows from the "Hypothetical Syllogism".

(iii) Let us consider the following simple propositions

p : Heera is a good

q : Sanjay is a good

In symbolic form

$p \vee q$

$\sim p$

$\therefore q$

Hence, $((p \vee q) \wedge \sim p) \to q$ is a tautology. This follows from ... Syllogism".

(iv) Let us consider the following simple propositions

p : polygon is a square

q : polygon has four sides

r : polygon has three angles

In symbolic form

$p \to q$

$q \to r$

$\therefore p \to r$

Hence, $((p \to q) \wedge (q \to r)) \to p \to r$ is a tautology. ... "Hypothetical Syllogism".

(v) Let us consider the following simple propositions

p : Bula is an excellent swimmer

q : Bula can work as a lifeguard

In symbolic form

p

$p \to q$

$\therefore q$

Hence, $(p \wedge (p \to q) \to q)$ is a tautology. This ... ponens".

(vi) Let us consider the following simple propositio...

p : I run fast

q : I will be champion in the race

In symbolic form

$p \to q$

$\sim q$

$\therefore \sim p$

Hence, $((p \to q) \wedge \sim q) \to \sim p$ is a tautology. T... tollens".

2. **Prove the validity of the following ar... rains, then it will be cold. If it is cold, t... rains therefore, I shall stay at home".**

Solution Let us consider the following simple...

In symbolic form

$p \vee q$

$-p$

$\therefore q$

Hence, $((p \vee q) \wedge -p) \rightarrow q$ is a tautology. This follows from the rule "Disjunctive Syllogism".

(iv) Let us consider the following simple propositions

p : polygon is a square

q : polygon has four sides

r : polygon has three angles

In symbolic form

$p \rightarrow q$

$q \rightarrow r$

$\therefore p \rightarrow r$

Hence, $((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow p \rightarrow r$ is a tautology. This follows from the rule "Hypothetical Syllogism".

(v) Let us consider the following simple propositions

p : Bula is an excellent swimmer

q : Bula can work as a lifeguard

In symbolic form

p

$p \rightarrow q$

$\therefore q$

Hence, $(p \wedge (p \rightarrow q) \rightarrow q)$ is a tautology. This follows from the rule "Modus ponens".

(vi) Let us consider the following simple propositions

p : I run fast

q : I will be champion in the race

In symbolic form

$p \rightarrow q$

$-q$

$\therefore -p$

Hence, $((p \rightarrow q) \wedge -q) \rightarrow -p$ is a tautology. This follows from the rule "Modus tollens".

2. Prove the validity of the following argument using truth tables "If it rains, then it will be cold. If it is cold, then I shall stay at home. Since, it rains therefore, I shall stay at home".

Solution Let us consider the following simple propositions

Solution (i) The truth table for $[(p \to q) \wedge (r \to s) \wedge (p \vee r)] \to (q \vee s)$ is

p	q	r	s	$p \to q$	$r \to s$	$p \vee r$	$(p \to q) \wedge (r \to s) \wedge (p \vee r)$	$[(p \to q) \wedge (r \to s) \wedge (p \vee r)] \to (q \vee s)$
T	T	T	T	T	T	T	T	T
T	T	T	F	T	F	T	F	F
T	T	F	T	T	T	T	T	T
T	T	F	F	T	T	T	T	T
T	F	T	T	F	T	T	T	T
T	F	T	F	F	F	T	F	F
T	F	F	T	F	T	T	T	T
T	F	F	F	F	T	T	F	F
F	T	T	T	T	T	T	T	T
F	T	T	F	T	F	T	F	F
F	T	F	T	T	T	T	T	T
F	T	F	F	T	T	F	F	F
F	F	T	T	T	T	T	T	T
F	F	T	F	T	F	T	F	F
F	F	F	T	T	T	F	F	F
F	F	F	F	T	T	F	F	F

We see all the truth values of $[(p \to q) \wedge (r \to s) \wedge (p \vee r)] \to (q \vee s)$ are "T". Hence this is a valid argument. Hence, $[(p \to q) \wedge (r \to s) \wedge (p \vee r)] \to (q \vee s)$ is a logical implication.

(ii) The truth table for $[(p \to q) \wedge (r \to s) \wedge (\neg q \vee \neg s)] \to (\neg p \vee \neg r)$ is

p	q	r	s	$p \to q$	$r \to s$	$\neg q \vee \neg s$	$\neg p \vee \neg r$	$[(p \to q) \wedge (r \to s) \wedge (\neg q \vee \neg s)] \to (\neg p \vee \neg r)$
T	T	T	T	T	T	F	F	T
T	T	T	F	T	F	T	F	T
T	T	F	T	T	T	F	T	T
T	T	F	F	T	T	T	T	T
T	F	T	T	F	T	T	F	T
T	F	T	F	F	F	T	F	T
T	F	F	T	F	T	T	T	T
T	F	F	F	F	T	T	T	T
F	T	T	T	T	T	F	T	T
F	T	T	F	T	F	T	T	T
F	T	F	T	T	T	F	T	T
F	T	F	F	T	T	T	T	T
F	F	T	T	T	T	F	T	T
F	F	T	F	T	F	T	T	T
F	F	F	T	T	T	F	T	T
F	F	F	F	T	T	T	T	T

We see all the truth values of $[(p \to q) \wedge (r \to s) \wedge (\neg q \vee \neg s)] \to (\neg p \vee \neg r)$ are "T". So, this is a valid argument. Hence, $[(p \to q) \wedge (r \to s) \wedge (\neg q \vee \neg s)] \to (\neg p \vee \neg r)$ is a logical implication.

6. **Prove that the argument $p \to (q \vee r)$, $(s \wedge t) \to q$, $(q \vee r) \to (s \wedge t)$ truth tables.**

Solution The three premises are : $p \to (q \vee r)$,

The conclusion is : $p \to q$

Now, the following sequence of elementary validity of the given argument:

(i) $p \to (q \vee r)$	Rule P (given, fir
(ii) $(s \wedge t) \to q$	Rule P (given, sec
(iii) $(q \vee r) \to (s \wedge t)$	Rule P (given, thi
(iv) $p \to (s \wedge t)$	Rule T (Hypothet
(v) $p \to q$	Rule T (Hypothet

Hence, the given argument is valid.

7. **Prove that the argument $(p \wedge q) \to r$, $r \to q$, $(r \wedge q) \to (q \wedge r)$ using truth tables.**

Solution The three premises are: $(p \wedge q) \to$

The conclusion is: $(p \wedge q) \to (q \wedge r)$

Now, the following sequence of element validity of the given argument :

(i) $(p \wedge q) \to r$	Rule P (given,
(ii) $r \to q$	Rule P (given,
(iii) $(r \wedge q) \to (q \wedge r)$	Rule P (given,
(iv) $r \to (r \wedge q)$	Rule T (rule of
(v) $(p \wedge q) \to (r \wedge q)$	Rule T (Hypot
(vi) $(p \wedge q) \to (q \wedge r)$	Rule T (Hypot

Hence, the given argument is valid.

8. **Prove the validity of the following**

(i) $p \vee q, \neg p \vdash q$

(ii) $p, p \to q, q \to r \vdash r$

(iii) $p, q, (p \wedge q) \to r \vdash r$

(iv) $p, (p \wedge \neg q) \to \neg\, p \vdash p \to q$

Solution (i) The two premises are : $p \vee$

The conclusion is : q

Now, the following sequence of elemen validity of the given argument :

6. **Prove that the argument $p \to (q \vee r), (s \wedge t) \to q, (q \vee r) \to (s \wedge t) \vdash p \to q$ is valid without using truth tables.**

Solution The three premises are : $p \to (q \vee r)$, $(s \wedge t) \to q$ and $(q \vee r) \to (s \wedge t)$

The conclusion is : $p \to q$

Now, the following sequence of elementary arguments is used to prove the validity of the given argument:

(i) $p \to (q \vee r)$	Rule P (given, first premise)
(ii) $(s \wedge t) \to q$	Rule P (given, second premise)
(iii) $(q \vee r) \to (s \wedge t)$	Rule P (given, third premise)
(iv) $p \to (s \wedge t)$	Rule T (Hypothetical syllogism using (i) and (iii))
(v) $p \to q$	Rule T (Hypothetical syllogism using (ii) and (iv))

Hence, the given argument is valid

7. **Prove that the argument $(p \wedge q) \to r, r \to q, (r \wedge q) \to (q \wedge r) \vdash (p \wedge q) \to (q \wedge r)$ is valid without using truth tables.**

Solution The three premises are: $(p \wedge q) \to r$, $r \to q$ and $(r \wedge q) \to (q \wedge r)$

The conclusion is: $(p \wedge q) \to (q \wedge r)$

Now, the following sequence of elementary arguments is used to prove the validity of the given argument :

(i) $(p \wedge q) \to r$	Rule P (given, first premise)
(ii) $r \to q$	Rule P (given, second premise)
(iii) $(r \wedge q) \to (q \wedge r)$	Rule P (given, third premise)
(iv) $r \to (r \wedge q)$	Rule T (rule of absorption using (ii))
(v) $(p \wedge q) \to (r \wedge q)$	Rule T (Hypothetical syllogism using (i) and (iv))
(vi) $(p \wedge q) \to (q \wedge r)$	Rule T (Hypothetical syllogism using (v) and (iii))

Hence, the given argument is valid

8. **Prove the validity of the following arguments without using truth tables:**

(i) $p \vee q, \sim p \vdash q$

(ii) $p, p \to q, q \to r \vdash r$

(iii) $p, q, (p \wedge q) \to r \vdash r$

(iv) $p, (p \wedge \sim q) \to \sim p \vdash p \to q$

Solution (i) The two premises are : $p \vee q$ and $\sim p$

The conclusion is : q

Now, the following sequence of elementary arguments is used to prove the validity of the given argument :

(i) $p \vee q$ Rule P (given, first premise)
(ii) $\sim p$ Rule P (given, second premise)
(iii) q Rule T (disjunctive syllogism)

Hence, the given argument is valid

(ii) $p, p \to q, q \to r \vdash r$

The three premises are : p, $p \to q$ and $q \to r$

The conclusion is: r

Now, the following sequence of elementary arguments is used to p validity of the given argument:

(i) p Rule P (given, first premise)
(ii) $p \to q$ Rule P (given, second premise)
(iii) $q \to r$ Rule P (given, third premise)
(iv) $p \to r$ Rule T (Hypothetical syllogism using (ii) and
(v) r Rule T (Modus ponens using (iv) and (i))

Hence, the given argument is valid

(iii) $p, q, (p \wedge q) \to r \vdash r$

The three premises are: p, q and $(p \wedge q) \to r$

The conclusion is : r

Now, the following sequence of elementary arguments is used to pro validity of the given argument :

(i) p Rule P (given, first premise)
(ii) q Rule P (given, second premise)
(iii) $(p \wedge q) \to r$ Rule P (given, third premise)
(iv) $p \wedge q$ Rule T (Rule of conjunction using (i) and (ii)
(v) r Rule T (Modus ponens using (iii)and (iv))

Hence, the given argument is valid

(iv) $p, (p \wedge \sim q) \to \sim p \vdash p \to q$

The two premises are : p and $(p \wedge \sim q) \to \sim p$

The conclusion is : $p \to q$

Now, the following sequence of elementary arguments is used to prov validity of the given argument:

(i) p Rule P [given, first premise]
(ii) $(p \wedge \sim q) \to \sim p$ Rule P [given, second premise]
(iii) $\sim (p \wedge \sim q)$ Rule T [Modus tollens using (ii) and (i)]
(iv) $\sim \sim (p \to q)$ Rule T [as $\sim (p \to q) \equiv (p \wedge \sim q)$]
(v) $(p \to q)$ Rule T [Complement property of (iv)]

Hence, the given argument is valid

9. **Prove the validity of the**
"If it does not rain or if
day will be held and th
day is held, the trophy
Therefore it rained".

Solution Let us consider the

p : It rains
q : There is traffic dislocatio
r : The sports day will be he
s : cultural programme will
t : The trophy will be awar
The first premise is $P_1 : \sim p$
The second premise is P_2
The third premise is $P_3 : \sim$
The conclusion is $Q : p$
Therefore, the argument
Therefore, the argument
$\sim p \vee \sim q \to r \wedge s, r \to t$
This argument will be va
$(\sim p \vee \sim q \to r \wedge s) \wedge (r$
Now, the following seq
validity of the given argum

(i) $\sim p \vee \sim q \to r \wedge s$
(ii) $r \to t$
(iii) $\sim t$
(iv) $(\sim p \to (r \wedge s) \wedge (\sim$
Rule CP
(v) $\sim (r \wedge s) \to p$
(vi) $\sim t \to \sim r$
(vii) $\sim r$
(viii) $\sim r \vee \sim s$
(ix) $\sim (r \wedge s)$
(x) p

Hence, the given argu

9. **Prove the validity of the following argument without using truth tables "If it does not rain or if there is no traffic dislocation, then the sports day will be held and the cultural programme will go on. If the sports day is held, the trophy will be awarded. The trophy was not awarded. Therefore it rained".**

Solution Let us consider the following simple propositions

p: It rains

q: There is traffic dislocation

r: The sports day will be held

s: cultural programme will go on

t: The trophy will be awarded

The first premise is $P_1 : \sim p \vee \sim q \rightarrow r \wedge s$

The second premise is $P_2 : r \rightarrow t$

The third premise is $P_3 : \sim t$

The conclusion is $Q : p$

Therefore, the argument is symbolically written as $P_1, P_2, P_3 \vdash Q$

Therefore, the argument is symbolically written as $P_1, P_2, P_3 \vdash Q$ that is

$\sim p \vee \sim q \rightarrow r \wedge s, r \rightarrow t, \sim t \vdash p$

This argument will be valid if $(P_1 \wedge P_2 \wedge P_3) \rightarrow Q$ that is

$(\sim p \vee \sim q \rightarrow r \wedge s) \wedge (r \rightarrow t) \wedge (\sim t) \rightarrow p$ is a tautology

Now, the following sequence of elementary arguments is used to prove the validity of the given argument :

(i)	$\sim p \vee \sim q \rightarrow r \wedge s$	Rule P (given, first premise)
(ii)	$r \rightarrow t$	Rule P (given, second premise)
(iii)	$\sim t$	Rule P (given, third premise)
(iv)	$(\sim p \rightarrow (r \wedge s) \wedge (\sim q \rightarrow (r \wedge s))$	Rule CP (equivalence rule $(a \vee b) \rightarrow c \equiv (a \rightarrow c) \wedge (b \rightarrow c)$)
(v)	$\sim (r \wedge s) \rightarrow p$	Rule T (contrapositive of (iv))
(vi)	$\sim t \rightarrow \sim r$	Rule T (contrapositive of (ii))
(vii)	$\sim r$	Rule T (Modus pones using (iii) and (vi))
(viii)	$\sim r \vee \sim s$	Rule T (addition of (vii))
(ix)	$\sim (r \wedge s)$	Rule T (Demorgan's law on (viii))
(x)	p	Rule T (Modus pones using (v) and (ix))

Hence, the given argument is valid

10. **Prove the validity of the following argument without using truth table: "Either I will pass the examination, or, will not graduate. If I graduate, I will go to USA. I failed; therefore I will go to USA."**

Solution Let us consider the following simple propositions

p : I will pass the examination

q : I will graduate

r : I will go to USA

The first premise is $P_1 : p \vee \sim q$

The second premise is $P_2 : \sim q \to r$

The third premise is $P_3 : \sim p$

The conclusion is $Q : r$

Therefore, the argument is symbolically written as $P_1, P_2, P_3 \vdash Q$

Therefore, the argument is symbolically written as $P_1, P_2, P_3 \vdash Q$ that is

$p \vee \sim q, \sim q \to r, \sim p \vdash r$

This argument will be valid if $(P_1 \wedge P_2 \wedge P_3) \to Q$ that is

$(p \vee \sim q) \wedge (\sim q \to r) \wedge \sim p \to r$ is a tautology

Now, the following sequence of elementary arguments is used to prove validity of the given argument:

(i)	$p \vee \sim q$	Rule P [given, first premise]
(ii)	$\sim q \to r$	Rule P [given, second premise]
(iii)	$\sim p$	Rule P [given, third premise]
(iv)	$\sim q$	Rule T [Disjunctive syllogism using (i) and (iii)]
(v)	r	Rule T [modus ponen using (ii) and (iv)]

Hence, the given argument is valid

11. **Prove the validity of the following argument without using truth table: "If I study, then I will pass examination. If I do not go to picnic, then I will study. But I failed examination. Therefore, I went to picnic."**

Solution Let us consider the following simple propositions

p : I will study

q : I will pass examination

r : I go to picnic

The first premise is $P_1 : p \to q$

The second premise is $P_2 : \sim r \to p$

The third premise is $P_3 : \sim q$

The conclusion is $Q : r$

Therefore, the argument is symbolically written as $P_1, P_2, P_3 \vdash Q$

Therefore, the argument is symbolically written as $P_1, P_2, P_3 \vdash Q$ that is

$p \to q, \sim r \to p, \sim q \vdash r$

This argument will be valid

$((p \to q) \wedge (\sim r \to p) \wedge (\sim$

Now, the following seque

validity of the given argumen

(i) $p \to q$

(ii) $\sim r \to p$

(iii) $\sim q$

(iv) $\sim p$

(v) $\sim(\sim r)$

(vi) r

Hence, the given argu

12. **Prove the validit is given by: "All is mortal"**

Solution Let us consi

x : Socrates

$P(x)$: x is a man

$H(x)$: x is mortal

With these symbolic

$\forall_x(P(x) \to Q(x))$

Now, the following

validity of the given ar

(i) $\forall_x (P(x) \to Q$

(ii) $P(x) \to Q(x)$

(iii) $P(x)$

(iv) $Q(x)$

Hence, the given

discussed in Chapte

Aliter : Let us co

p : Socrates

q : Man

r : Mortal

In symbolic form

$p \to q$: Socrates

$q \to r$: All men

$p \to r$: Socrates

Hence, the argu

"Hypothetical syll

This argument will be valid if $(P_1 \wedge P_2 \wedge P_3) \rightarrow Q$ that is
$((p \rightarrow q) \wedge (\neg r \rightarrow p) \wedge (\neg q)) \rightarrow r$ is a tautology

Now, the following sequence of elementary arguments is used to prove the validity of the given argument:

(i) $p \rightarrow q$	Rule P [given, first premise]
(ii) $\neg r \rightarrow p$	Rule P [given, second premise]
(iii) $\neg q$	Rule P [given, third premise]
(iv) $\neg p$	Rule T [Modus tollens using (i) and (iii)]
(v) $\neg(\neg r)$	Rule T [Modus tollens using (ii) and (iv)]
(vi) r	Rule T [Complement property using (v)]

Hence, the given argument is valid

12. Prove the validity of the following "Famous Socrates argument" which is given by: "All men are mortal. Socrates is a man. Therefore, Socrates is mortal"

Solution Let us consider the following notations

x: Socrates

$P(x)$: x is a man

$H(x)$: x is mortal

With these symbolic notations, the mathematical statement is

$\forall_x(P(x) \rightarrow Q(x)) \wedge P(x) \rightarrow Q(x)$

Now, the following sequence of elementary arguments is used to prove the validity of the given argument:

(i) $\forall_x (P(x) \rightarrow Q(x))$	Rule P [given]
(ii) $P(x) \rightarrow Q(x)$	Rule P [given]
(iii) $P(x)$	Rule P [given]
(iv) $Q(x)$	Rule T [Modus ponens using (ii) and (iii)]

Hence, the given argument is valid. [This proof follows from Predicate logic discussed in Chapter 3]

Aliter : Let us consider the simple propositions

p: Socrates

q: Man

r: Mortal

In symbolic form

$p \rightarrow q$: Socrates is a man (first premise)

$q \rightarrow r$: All men are mortal (second premise)

$p \rightarrow r$: Socrates is mortal (conclusion)

Hence, the argument $(p \rightarrow q) \wedge (q \rightarrow r) \rightarrow (p \rightarrow r)$ is valid argument by the rule "Hypothetical syllogism".

13. **Give a direct proof of the statement "The Square of an even integer is an even integer".**

Solution Let p and q be two simple propositions. Let x be an even integer.

$x = 2n, n \in \mathbb{Z}$ (set of integers)

p : the square of even integer x

q : x is even integer

Squaring both sides,

$x^2 = (2n)^2 = 4n^2 = 2(2n^2) = 2t$, where $t = 2n^2 \in \mathbb{Z}$

This is even. Thus, the square of an even integer is an even integer.

Therefore, p is true and $p \rightarrow q$ is true, we conclude that q is true by the "Modus ponens".

14. **Give an indirect proof of the statement "If $3n + 2$ is odd, then n**

Solution Let p and q be two simple propositions.

p : $3n + 2$ is odd

q : n is odd

We have to prove $p \rightarrow q$. We shall prove the statement by using its equ contrapositive statement : $\sim q \rightarrow \sim p$.

Now, $\sim q$: n is even

$\Rightarrow n = 2t, t \in \mathbb{Z}$

Consider, $3n + 2 = 3(2t) + 2 = 6t + 2 = 2(3t + 1)$

This is even

$\Rightarrow 3n + 2$ is even $\Rightarrow \sim p$ holds.

Therefore, by the rule of contrapositive the result holds.

Aliter: This statement may also be proved by the method of contradict

Let p and q be two simple propositions.

p : $3n + 2$ is odd

q : n is odd

Let $\sim q$ is true that is n is even,

Then $n = 2t, t \in \mathbb{Z}$ (set of integers)

It follows that $3n + 2 = 3(2t) + 2 = 6t + 2 = 2(3t + 1)$

$\Rightarrow 3n + 2$ is even (since it can be expressed as a multiple of 2)

We arrive at a contradiction that p : $3n + 2$ is even but the hypothesis is odd. This contradiction proves that q is true and the implication $p \rightarrow q$ is

15. **Prove that $\sqrt{5}$ is irrational by contradiction method.**

Solution Let p and q be two simple propositions.

p : x and y are positive integers

q : $\sqrt{5}$ is irrational

Let us consider $\sqrt{5}$ is rational (~q

Then, $\frac{x}{y} = \sqrt{5}$, x and y are relati

This implies $x = \sqrt{5}\, y$

$\Rightarrow x^2 = 5y^2$

$\Rightarrow 5 \mid x^2$ (5 divides x^2)

$\Rightarrow 5 \mid x$ (5 divides x)

Therefore, by definition, $x = 5t$

$x^2 = 25t^2$

But, $x^2 = 5y^2$

So, $5y^2 = 25t^2$

$\Rightarrow y^2 = 5t^2$

$\Rightarrow 5 \mid y^2$ (5 divides y^2)

$\Rightarrow 5 \mid y$ (5 divides y)

Equations (i) and (ii) implies t assumption that x and y are rel rational is wrong. So, $\sqrt{5}$ is irrati

16. **Using Mathematical In 3, for all $n \in$ N**

Solution Let P(n) be the math

$10^{n+1} + 10^n + 1$ is divisible

Basic step :

For $n = 1$; $10^{n+1} + 10^n + 1 = 1$

$\therefore$ P(n) is true for $n = 1$

Inductive step :

Let P(n) is true for $n = k$

i.e. $10^{k+1} + 10^k + 1$ is divisibl

So, $10^{k+1} + 10^k + 1 = 3m$, for

Now, $[10^{(k+1)+1} + 10^{k+1} + 1$

$= 10 \cdot 10^{k+1} + 10 \cdot 10^k$

$= 9 \cdot 10^{k+1} + 9 \cdot 10^k = 3$

i.e., $10^{k+2} + 10^{k+1} + 1 = 3m$

$= 3(m$

This proves that P(n) is also

$\therefore$ By the first principle of i

Let us consider $\sqrt{5}$ is rational ($\sim q$ is true).

Then, $\frac{x}{y} = \sqrt{5}$, x and y are relatively prime.

This implies $x = \sqrt{5}\, y$

$\Rightarrow x^2 = 5y^2$

$\Rightarrow 5 \mid x^2$ (5 divides x^2)

$\Rightarrow 5 \mid x$ (5 divides x)

Therefore, by definition, $x = 5t$ for some $t \in \mathbb{Z}$. ... (i)

$x^2 = 25t^2$

But, $x^2 = 5y^2$

So, $5y^2 = 25t^2$

$\Rightarrow y^2 = 5t^2$

$\Rightarrow 5 \mid y^2$ (5 divides y^2)

$\Rightarrow 5 \mid y$ (5 divides y) ...(ii)

Equations (i) and (ii) implies that 5 divides both x and y which contradicts our assumption that x and y are relatively prime. Thus, our assumption that $\sqrt{5}$ is rational is wrong. So, $\sqrt{5}$ is irrational

16. Using Mathematical Induction show that $10^{n+1} + 10^n + 1$ is divisible by 3, for all $n \in \mathbb{N}$

Solution Let P(n) be the mathematical statement :

$10^{n+1} + 10^n + 1$ is divisible by 3.

Basic step :

For $n = 1$; $10^{n+1} + 10^n + 1 = 10^2 + 10 + 1 = 111$, which is divisible by 3.

$\therefore$ P(n) is true for $n = 1$

Inductive step :

Let P(n) is true for $n = k$

i.e. $10^{k+1} + 10^k + 1$ is divisible by 3

So, $10^{k+1} + 10^k + 1 = 3m$, for some $m \in \mathbb{Z}$

Now, $[10^{(k+1)+1} + 10^{k+1} + 1] - [10^{k+1} + 10^k + 1]$

$= 10 \cdot 10^{k+1} + 10 \cdot 10^k - 10^{k+1} - 10^k$

$= 9 \cdot 10^{k+1} + 9 \cdot 10^k = 3[3 \cdot 10^{k+1} + 3 \cdot 10^k]$

i.e., $10^{k+2} + 10^{k+1} + 1 = 3m + 3[3 \cdot 10^{k+1} + 3 \cdot 10^k]$ [since $10^{k+1} + 10^k + 1 = 3m$]

$= 3[m + \{3 \cdot 10^{k+1} + 3 \cdot 10^k\}]$, which is divisible by 3.

This proves that P(n) is also true for $n = k + 1$, whenever it is true for $n = k$.

$\therefore$ By the first principle of induction, P(n) is true for all $n \in \mathbb{N}$

17. **Using Mathematical Induction show $7^n - 1$ is divisible by 6 for all $n \geq 0$.**

Solution For $n = 0$, $7^0 - 1 = 0$, which is divisible by 6

So, our result holds for $n = 0$.

Let P(n) be the mathematical statement :

$7^n - 1$ is divisible by 6 for all $n \in \mathbb{N}$

Basic step :

Again for $n = 1$, $7^1 - 1 = 6$, which is divisible by 6.

$\therefore$ P(n) is true for $n = 1$.

Inductive step :

Let P(n) is true for $n = k$

i.e., $7^k - 1$ is divisible by 6.

So, $7^k - 1 = 6m$, where $m \in \mathbb{Z}$

Now, $[7^{k+1} - 1] - [7^k - 1] = 7 \cdot 7^k - 7^k = 6 \cdot 7^k$

i.e., $7^{k+1} - 1 = 6 \cdot 7^k + [7^k - 1] = 6.7^k + 6m$ [since, $7^k - 1 = 6m$]

$= 6(7^k + m)$,

which is divisible by 6.

This proves that P(n) is also true for $n = k + 1$, whenever it is true for $n = k$

$\therefore$ By the first principle of induction, P(n) is true for all $n \in \mathbb{N}$.

Again, we already establish that our result also holds for $n = 0$. Hence, we conclude that, $7^n - 1$ is divisible by 6 for all integer $n \geq 0$.

18. **Using Mathematical Induction show that 19 divides $7^{n+2} + 8^{2n+1}$ for integers $n \geq 1$.**

Solution Let P(n) be the mathematical statement :

19 divides $7^{n+2} + 8^{2n+1}$ for all integer $n \geq 1$

Basic step :

For $n = 1$; $7^{1+2} + 8^{2 \cdot 1 + 1} = 7^3 + 8^3 = 855$

19 divides 855.

$\therefore$ P(n) is true for $n = 1$.

Inductive step :

Let P(n) is true for $n = k$

i.e., 19 divides $7^{k+2} + 8^{2k+1}$.

i.e., $7^{k+2} + 8^{2k+1} = 19m$; where $m \in \mathbb{Z}$.

Now, $[7^{(k+1)+2} + 8^{2(k+1)+1}] - [7^{k+2} + 8^{2k+1}]$

$= [7 \cdot 7^{k+2} + 64 \cdot 8^{2k+1}] - [7^{k+2} + 8^{2k+1}]$

$= 6 \cdot 7^{k+2} + 63 \cdot 8^{2k+1}$

$= 6 \cdot 7^{k+2} + 6 \cdot 8^{2k+1} + 57 \cdot 8^{2k+1}$

$= 6(7^{k+2} + 8^{2k+1}) + 19$

$= 6 \cdot 19m + 19 \cdot 3 \cdot 8^{2k+1}$

$= 19(6m + 3 \cdot 8^{2k+1})$

i.e., $[7^{(k+1)+2} + 8^{2(k+1)+1}$

$\therefore$ 19 divides $19(7m + 3$

This proves that P(n) is

$\therefore$ By the first principle

19. **Using Mathematic all $n \in \mathbb{N}$.**

Solution Let P(n) be the

$1 + 2 + 3 + \ldots + n = \frac{n}{}$

Basic step :

For $n = 1$, L.H.S. = 1 and

$\therefore$ P(n) is true for $n = 1$.

Inductive step :

Let P(n) is true for $n = k$

i.e., $1 + 2 + 3 + \ldots + k = \frac{k}{}$

Now, $1 + 2 + 3 + \ldots + k +$

$= \frac{k(k+1)}{2} + (k +$

$= (k + 1)\left(\frac{k}{2} + 1\right) =$

This proves that P(n) is

$\therefore$ By the first principle

20. **Using Mathematic**

$3 + 3.5 + 3.5$

Solution Let P(n) be the

$3 + 3.5 + 3.5^2 + \ldots + 3.5$

Basic step :

For $n = 1$, L.H.S. = 3 and

$\therefore$ P(n) is true for $n = 1$.

$= 6(7^{k+2} + 8^{2k+1}) + 19 \cdot 3 \cdot 8^{2k+1}$

$= 6 \cdot 19m + 19 \cdot 3 \cdot 8^{2k+1}$ [$\because 7^{k+2} + 8^{2k+1} = 19m$]

$= 19(6m + 3 \cdot 8^{2k+1})$

i.e., $7^{(k+1)+2} + 8^{2(k+1)+1}] = 7^{k+2} + 8^{2k+1} + 19(6m + 3 \cdot 8^{2k+1})$ [$\because 7^{k+2} + 8^{2k+1} = 19m$]

$= 19m + 19(6m + 3 \cdot 8^{2k+1})$

$= 19(7m + 3 \cdot 8^{2k+1})$

$\therefore$ 19 divides $19(7m + 3 \cdot 8^{2k+1})$.

This proves that P(n) is true for $n = k + 1$, whenever it is true for $n = k$.

$\therefore$ By the first principle of induction, P(n) is true for all integers $n \geq 1$

19. **Using Mathematical Induction show that $1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$ for all $n \in \mathbb{N}$.**

Solution Let P(n) be the mathematical statement :

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$$

Basic step :

For $n = 1$, L.H.S. $= 1$ and R.H.S. $= \frac{1 \cdot 2}{2} = 1$.

$\therefore$ P(n) is true for $n = 1$.

Inductive step :

Let P(n) is true for $n = k$.

i.e., $1 + 2 + 3 + \ldots + k = \frac{k(k+1)}{2}$

Now, $1 + 2 + 3 + \ldots + k + (k + 1)$

$= \frac{k(k+1)}{2} + (k + 1)$ [By inductive hypothesis]

$= (k + 1)\left(\frac{k}{2} + 1\right) = \frac{(k+1)(k+2)}{2}$

This proves that P(n) is also true for $n = k + 1$; whenever it is true for $n = k$.

$\therefore$ By the first principle of induction, P(n) is true for all n.

20. **Using Mathematical Induction show that**

$$3 + 3.5 + 3.5^2 + \cdots + 3.5^{n-1} = \frac{3}{4}(5^n - 1); \forall\, n \in \mathbb{Z}^+.$$

Solution Let P(n) be the mathematical statement:

$$3 + 3.5 + 3.5^2 + \cdots + 3.5^{(n-1)} = \frac{3}{4}(5^n - 1)$$

Basic step :

For $n = 1$, L.H.S. $= 3$ and R.H.S. $= \frac{3}{4}(5 - 1) = 3$.

$\therefore$ P(n) is true for $n = 1$.

Inductive step :

Let P(n) is true for $n = k$.

i.e., $3 + 3.5 + 3.5^2 + \cdots + 3.5^{k-1} = \frac{3}{4}(5^k - 1)$.

Now, $3 + 3.5 + 3.5^2 + \cdots + 3.5^{(k-1)} + 3.5^k$

$= \frac{3}{4}(5^k - 1) + 3.5^k$ [By inductive hypothesis]

$= \frac{3}{4}(5^k - 1 + 4.5^k) = \frac{3}{4}(5.5^k - 1) = \frac{3}{4}(5^{k+1} - 1)$

This proves that P(n) is also true for $n = k + 1$; whenever it is true for

$\therefore$ By the first principle of induction, P(n) is true, $\forall n \in \mathbb{Z}^+$.

21. **Using Mathematical Induction show that $4^{n+1} + 5^{2n-1}$ is divisi for all $n \in \mathbb{Z}^+$**

Solution Let P(n) be the mathematical statement :

$4^{n+1} + 5^{2n-1}$ is divisible by 21 for all $n \in \mathbb{Z}^+$.

Basic step :

For $n = 1$; $4^{1+1} + 5^{2.1-1} = 4^2 + 5 = 21$, which is divisible by 21.

$\therefore$ P(n) is true for $n = 1$.

Inductive step :

Let P(n) is true for $n = k$.

i.e., $4^{k+1} + 5^{2k-1}$ is divisible by 21.

So, $f(k) = 21m$; where $m \in \mathbb{Z}$ and $f(k) = 4^{k+1} + 5^{2k-1}$

Now $f(k+1) - f(k) = [4^{(k+1)+1} + 5^{2(k+1)-1}] - [4^{k+1} + 5^{2k-1}]$

$= 4.4^{k+1} + 5^{2k-1}.5^2 - 4^{k+1} - 5^{2k-1} = (4-1).4^{k+1} + (25-1)$

$= 3(4^{k+1} + 5^{2k-1}) + 21.5^{2k-1} = 3.21m + 21.5^{2k-1}$ [$\because f(k$

$= 21(3m + 5^{2k-1})$

i.e., $f(k+1) = f(k) + 21(3m + 5^{2k-1}) = 21m + 21(3m + 5^{2k-1})$

$= 21(4m + 5^{2k-1})$, which is divisible by 21.

This proves that P(n) is true for $n = k + 1$, whenever it is true $n = k$

$\therefore$ By the first principle of induction, P(n) is true for all $n \in \mathbb{Z}^+$

22. **Using Mathematical Induction show that $3^n + 7^n - 2$ is divisible all $n \in \mathbb{Z}^+$**

Solution Let P(n) be the mathematical statement :

$3^n + 7^n - 2$ is divisible by 8.

Basic step :

For $n = 1$; $3^1 + 7^1 - 2 = 8$, which is divisible by 8.

$\therefore$ P(n) is true for $n = 1$.

Inductive step :

Let P(n) is true for $n = k$.

i.e., $3^k + 7^k - 2$ is divisible by 8.

i.e., $f(k) = 8m$; where $m \in \mathbb{Z}$ and $f(k) = 3$

Now, $f(k+1) - f(k) = [3^{k+1} + 7^{k+1} - 2] - [$

$= 3.3^k + 7.7^k - 3^k - 7^k = 2.3^k + 6.7$

$= 2.(3^k + 7^k - 2) + 4 + 4.7^k = 2.(3$

$= 2.8m + 4.(2p)$

$= 8(2m + p)$

i.e., $f(k+1) = 8(2m + p) + f(k) = 8(2m$

$= 8(3m + p)$, which is divi

This proves that P(n) is true for $n = k$

$\therefore$ By the first principle of induction,

23. **Using Mathematical Inductio $\forall n \in \mathbb{Z}^+$.**

Solution Let P(n) be the mathematic

$3^{2n+1} + (-1)^n 2 \equiv 0 \pmod 5$

i.e., $3^{2n+1} + (-1)^n 2$ is divisible by 5.

Basic step :

For $n = 1$; $3^{2.1+1} + 2.(-1)^1 = 3^3 - 2 = 2$

$\therefore$ P(n) is true for $n = 1$.

Inductive step :

Let P(n) is true for $n = k$.

i.e., $[3^{2k+1} + (-1)^k.2] \equiv 0 \pmod 5$

i.e., $f(k)$ is divisible by 5, where $f(k)$

i.e., $f(k) = 5m$; where $m \in \mathbb{Z}^+$.

Now $f(k+1) - f(k) = [3^{2(k+1)+1} + (-1)^$

$= 9.3^{2k+1} + (-1)^{k+1}.2 - 3^{2k+1} + (-1)$

$= 8.(3^{2k+1} + (-1)^k.2) - 16.(-1)^k - ($

$= 8.5m - 20.(-1)^k$ [By the hypoth

$= 5.[8m - 4.(-1)^k]$

i.e., $f(k+1) = 5[8m - 4.(-1)^k] + f(k)$

which is divisible by 5, then $3^{2(k$

This proves that P(n) is true for n

$\therefore$ By the first principle of induction

Inductive step :

Let P(n) is true for $n = k$.

i.e., $3^k + 7^k - 2$ is divisible by 8.

i.e., $f(k) = 8m$; where $m \in \mathbb{Z}$ and $f(k) = 3^k + 7^k - 2$

Now, $f(k+1) - f(k) = [3^{k+1} + 7^{k+1} - 2] - [3^k + 7^k - 2]$

$= 3.3^k + 7.7^k - 3^k - 7^k = 2.3^k + 6.7^k = 2.(3^k + 7^k) + 4.7^k$

$= 2.(3^k + 7^k - 2) + 4 + 4.7^k = 2.(3^k + 7^k - 2) + 4(1 + 7^k)$

$= 2.8m + 4.(2p)$ [By the hypothesis $f(k) = 8m$ and $p \in \mathbb{Z}$, also $1 + 7^k$ is divisible by 2 for all $k \in \mathbb{Z}^+$]

$= 8(2m + p)$

i.e., $f(k+1) = 8(2m + p) + f(k) = 8(2m + p) + 8m$

$= 8(3m + p)$, which is divisible by 8.

This proves that P(n) is true for $n = k + 1$; whenever it is true for $n = k$.

∴ By the first principle of induction, P(n) is true for all integers $n \in \mathbb{Z}^+$.

23. **Using Mathematical Induction show that $3^{2n+1} + (-1)^n 2 \equiv 0 \pmod 5$; $\forall n \in \mathbb{Z}^+$.**

Solution Let P(n) be the mathematical statement :

$3^{2n+1} + (-1)^n 2 \equiv 0 \pmod 5$

i.e., $3^{2n+1} + (-1)^n 2$ is divisible by 5.

Basic step :

For $n = 1$; $3^{2.1+1} + 2.(-1)^1 = 3^3 - 2 = 25$

∴ P(n) is true for $n = 1$.

Inductive step :

Let P(n) is true for $n = k$.

i.e., $[3^{2k+1} + (-1)^k.2] \equiv 0 \pmod 5$

i.e., $f(k)$ is divisible by 5, where $f(k) = 3^{2k+1} + (-1)^k.2$

i.e., $f(k) = 5m$; where $m \in \mathbb{Z}^+$.

Now $f(k+1) - f(k) = [3^{2(k+1)+1} + (-1)^{k+1}.2] - [3^{2k+1} + (-1)^k.2]$

$= 9.3^{2k+1} + (-1)^{k+1}.2 - 3^{2k+1} + (-1)^{k+1}.2 = 8.3^{2k+1} - 4.(-1)^k$

$= 8.(3^{2k+1} + (-1)^k.2) - 16.(-1)^k - (-1)^k.4$

$= 8.5m - 20.(-1)^k$ [By the hypothesis]

$= 5.[8m - 4.(-1)^k]$

i.e., $f(k+1) = 5[8m - 4.(-1)^k] + f(k) = 5[8m - 4.(-1)^k] + 5m = 5[9m - 4.(-1)^k]$

which is divisible by 5, then $3^{2(k+1)+1} + (-1)^{k+1}.2 \equiv 0 \pmod 5$

This proves that P(n) is true for $n = k + 1$, whenever it is true for $n = k$.

∴ By the first principle of induction, P(n) is true for all $n \in \mathbb{Z}^+$

2.16 Adequate set of Connectives

Let S be a set of connectives. If the set of connectives S is sufficient to ... every possible propositional formula, we call it an ***adequate set of connectives*** ... functionally complete set of connectives.

If S does not contain a connective which can be expressed in ... other connectives of S, then S is called ***minimal functionally*** complete ... connectives. So, an adequate set of connectives is ***minimal*** if no proper ... of it is adequate.

Example 2.30 : The set $\{\sim, \wedge\}$ and $\{\sim, \vee\}$ are minimal functionally complete ... connectives. The conditional and biconditional connectives can be written ...

$p \to q \equiv \sim p \vee q$ and $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$

Now, $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$

$\equiv (\sim p \vee q) \wedge (\sim q \vee p)$

Also, by De Morgan's law, we can write

$(p \vee q) \equiv \sim(\sim p \wedge \sim q)$ and, $(p \wedge q) \equiv \sim(\sim p \vee \sim q)$

Thus, we can express any propositional formula or statement formula ... contains the logical operators $\wedge$, $\vee$, $\to$ or $\leftrightarrow$ by another statement formula ... contains only the set of connectives $\{\sim, \wedge\}$ or $\{\sim, \vee\}$.

So, the set $\{\sim, \wedge\}$ and $\{\sim, \vee\}$ are minimal functionally complete set of connectives.

Example 2.31 : Now the question is whether the set $\{\vee, \wedge, \sim\}$ is minimal ... The answer is "no" because, by De Morgan's Laws both $\{\wedge, \sim\}$ and $\{\vee, \sim\}$ ... adequate sets of connectives and $\{\wedge, \sim\}$ and $\{\vee, \sim\}$ are proper subsets of ...

2.17 Normal Forms

To determine whether a compound propositions $P = f_1(p_1, p_2, \ldots, p_n)$ ... $p_1, p_2, \ldots, p_n$ are simple propositions, is a tautology or contradiction or ... **satisfiable** and whether two given compound propositions $P = f_1(p_1, p_2, \ldots$ and $Q = f_2(p_1, p_2, \ldots, p_n)$ are equivalent, we have to construct the truth tables ... compare them.

The problem of determining in a finite number of steps whether a ... statement formula is a tautology or a contradiction or at least satisfiable is known ... as a decision problem. The decision problem always has a solution.

But the construction of the truth values may not be practical when ... number of primary propositions (variables) $p_1, p_2, \ldots, p_n$ increases. A better ... method is to reduce $P = f_1(p_1, p_2, \ldots, p_n)$ and $Q = f_2(p_1, p_2, \ldots, p_n)$ to ... standard forms, known as *Normal forms* and use them for deciding the nature ... $P = f_1(p_1, p_2, \ldots, p_n)$ and $Q = f_2(p_1, p_2, \ldots, p_n)$ and for comparing $P = f_1(p_1, p_2, \ldots$ $Q = f_2(p_1, p_2, \ldots, p_n)$. There are two types of normal forms— **Disjunctive normal form** (DNF) and **Conjunctive normal form** (CNF).

Notes

1. A compound proposition $P = f_1(p_1, p_2, \ldots, p_n)$ is ... a truth value T for at least one combination of the ...
2. The word 'product' can be used in place of 'con... can be used in place of 'disjunction'.

2.17.1 Disjunctive Normal Form

By a *literal* we mean a statement variable ... variable $\sim p$.

A statement form is said to be in *disjunctive* ... disjunction consisting of one or more disjuncts ... of one or more literals. Thus, compound propositions ... to be in Disjunctive Normal (DNF) form if P can be ...

$$P = P_1 \vee P_2 \vee \ldots \vee P_n$$

Where each P_i is the conjunction of some ... p_n or their negations without any repetition ... $P_j (j \neq i)$. DNF can also be defined as the sum of ... propositions. DNF of a given formula is not unique.

Example 2.32 : $(p_1 \wedge \sim p_3) \vee (\sim p_2 \wedge p_1 \wedge p_3)$ ... propositions p_1, p_2, p_3.

Example 2.33 : $(p_1 \wedge - p_3) \vee (p_1 \wedge \sim p_2 \wedge p_3)$ ... $(p_1 \wedge \sim p_3)$ is contained in $(\sim p_3 \wedge p_1 \wedge p_2)$

2.17.2 Conjunctive Normal Form

A statement form is in *conjunctive normal* ... one or more conjuncts, each of which is a ... compound propositions $P = f_1(p_1, p_2, \ldots, p_n)$ ... (CNF) form if P can be expressed as

$$P = P_1 \wedge P_2 \wedge \ldots \wedge P_n$$

where each P_i is the disjunction of finite ... each of $p_1, p_2, \ldots, p_n$ or their negations and ...

CNF can also be defined as the product ... propositions. CNF of a given formula is ... identically true if every elementary sum is ...

Example 2.34 : $(\sim p_1 \vee p_2 \vee p_3) \wedge (\sim p_1 \vee \ldots$ ... the simple propositions p_1, p_2, p_3.

Example 2.35 : $(\sim p_1 \vee p_2) \wedge (\sim p_1 \vee - p_2$ ... not contain p_3 or its negation

Example 2.36 : $p_1 \wedge (\sim p_1 \vee \sim p_2)$ is not ...

Notes

1. A compound proposition $P = f_1(p_1, p_2, ..., p_n)$ is said to be *satisfiable*, if it has a truth value T for at least one combination of the truth values of $p_1, p_2, ..., p_n$
2. The word 'product' can be used in place of 'conjunction' and the word 'sum' can be used in place of 'disjunction'.

2.17.1 Disjunctive Normal Form

By a *literal* we mean a statement variable p or a negation of a statement variable $\sim p$.

A statement form is said to be in *disjunctive normal form* (DNF) if it is a disjunction consisting of one or more disjuncts, each of which is a conjunction of one or more literals. Thus, compound propositions $P = f_1(p_1, p_2, ..., p_n)$ is said to be in Disjunctive Normal (DNF) form if P can be expressed as

$$P = P_1 \vee P_2 \vee ... \vee P_n$$

Where each P_i is the conjunction of some simple propositions $p_1, p_2, ..., p_n$ or their negations without any repetitions and no P_i is contained in any $P_j (j \neq i)$. DNF can also be defined as the sum of elementary products of the simple propositions. DNF of a given formula is not unique.

Example 2.32 : $(p_1 \wedge \sim p_3) \vee (\sim p_2 \wedge p_1 \wedge p_3) \vee (\sim p_1 \wedge \sim p_2)$ is a DNF of the simple propositions p_1, p_2, p_3.

Example 2.33 : $(p_1 \wedge - p_3) \vee (p_1 \wedge \sim p_2 \wedge p_3) \vee (\sim p_3 \wedge p_1 \wedge p_2)$ is not a DNF since $(p_1 \wedge \sim p_3)$ is contained in $(\sim p_3 \wedge p_1 \wedge p_2)$.

2.17.2 Conjunctive Normal Form

A statement form is in *conjunctive normal form* (CNF) if it is a conjunction of one or more conjuncts, each of which is a disjunction of one or more literals. A compound propositions $P = f_1(p_1, p_2, ..., p_n)$ is said to be in Conjunctive Normal (CNF) form if P can be expressed as

$$P = P_1 \wedge P_2 \wedge ... \wedge P_n$$

where each P_i is the disjunction of finite number of the simple propositions of each of $p_1, p_2, ..., p_n$ or their negations and $P_i \neq P_j$ for all distinct i, j.

CNF can also be defined as the product of elementary sums of the simple propositions. CNF of a given formula is not unique. For a given formula CNF is identically true if every elementary sum is identically true.

Example 2.34 : $(\sim p_1 \vee p_2 \vee p_3) \wedge (\sim p_1 \vee \sim p_2 \vee \sim p_3) \wedge (p_1 \vee \sim p_2 \vee p_3)$ is a CNF of the simple propositions p_1, p_2, p_3.

Example 2.35 : $(\sim p_1 \vee p_2) \wedge (\sim p_1 \vee - p_2 \vee p_3)$ is not a CNF since $(\sim p_1 \vee p_2)$ does not contain p_3 or its negation

Example 2.36 : $p_1 \wedge (\sim p_1 \vee \sim p_2)$ is not a CNF

Example 2.37 : $(p_1 \vee \sim p_2) \wedge (\sim p_1 \vee p_2)$ is a CNF of the simple prop... p_1, p_2.

A DNF (CNF) is called *full* if no disjunct (conjunct) contains two occu... of literals with the same variable and if a variable that occurs in one ... (conjunct) also occurs in all the others. A CNF is said to be a *blatant contra...* if it contains some variable and its negation as conjuncts.

Example 2.38 : $(\sim p_1 \wedge p_2 \wedge p_3) \vee (\sim p_1 \wedge \sim p_2 \wedge \sim p_3) \vee (p_1 \wedge \sim p_2 \wedge p_3)$ ... DNF while $(p_1 \wedge \sim p_3) \vee (\sim p_2 \wedge p_1 \wedge p_3) \vee (\sim p_1 \wedge \sim p_2)$ is not.

Example 2.39 : $(p_1) \wedge (p_2 \vee \sim p_3) \wedge (p_2 \vee p_3) \wedge (\sim p_1)$ is a *blatant contra...* as it contains both p_1 and $\sim p_1$ as conjuncts.

Notes

1. The dual notion of CNF is DNF.
2. A compound proposition 'f' can be put in CNF or DNF using the la... Algebra of propositions (Section 2.6).

2.18 Minterms and Maxterms

Given a number of variables, the conjunctions in which each variable ... negation, but not both, occur only once are called the *minterms*. For two va... p and q, the possible minterms are $p \wedge q$, $p \wedge \sim q$, $\sim p \wedge q$ and $\sim p \wedge \sim q$.

For three variables p, q and r, the possible minterms are $p \wedge q \wedge r$, $\sim p \wedge$ $p \wedge \sim q \wedge r$, $p \wedge q \wedge \sim r$, $\sim p \wedge \sim q \wedge r$, $p \wedge \sim q \wedge \sim r$, $\sim p \wedge q \wedge \sim r$, $\sim p \wedge \sim q \wedge \sim r$. Ther... 2^n minterms for n variables.

Given a number of variables, the disjunctions in which each variab... its negation, but not both, occur only once are called the *maxterms*. For ... variables p and q, the possible maxterms are $p \vee q$, $p \vee \sim q$, $\sim p \vee q$ and $\sim p$... The maxterms are simply the duals of the minterms.

For three variables p, q and r, the possible maxterms are $p \vee q \vee r$, $\sim p \vee$ $p \vee \sim q \vee r$, $p \vee q \vee \sim r$, $\sim p \vee \sim q \vee r$, $p \vee \sim q \vee \sim r$, $\sim p \vee q \vee \sim r$, $\sim p \vee \sim q \vee \sim r$. Ther... 2^n maxterms for n variables.

Note

The minterms and maxterms are dual to each other

2.19 Principal Disjunctive and Conjunctive Normal Fo...

A compound proposition consisting of disjunctions of minterms o... variables only and equivalent to a given formula is known as its prin... disjunctive normal form (PDNF). PDNF is also called 'sum of product canon... form'.

Example 2.40 : $(p \wedge q \wedge r) \vee (p \wedge q \wedge \sim r$... of the simple propositions p, q, r.

Similarly, a compound proposition ... in the variables only and equivale... principal conjunctive normal form ... sum canonical form'.

Example 2.41 : $(p \vee q \vee r) \wedge (p \vee q \vee$... of the simple propositions p, q, r.

Note

If the PDNF of a formula 'f' is kno... disjunctions of the remaining minte... of 'f'.

To obtain the PCNF of 'f', we use the ... laws to the PDNF of '$\sim f$' repeatedly.

2.20 Construction of PD...

If 'f' is a compound proposition, t...

- **Step-1.** Build the truth table of...
- **Step-2.** For each row that ... table, generate a conjunction ... row.
- **Step 3.** Take the disjunction o...

Example 2.42 : Let us consider ...

Table

p
T
T
F
F

In the last column of the above ... row. The conjunctions of **literals** (... $(p \wedge q)$ and $(\sim p \wedge \sim q)$. Now the disj... $(\sim p \wedge \sim q)$ is $(p \wedge q) \vee (\sim p \wedge \sim q)$, wh...

If 'f' is a compound proposition ...

- **Step 1.** Build the truth table ...
- **Step 2.** Consider each row th... Take the simple proposition ... propositions for "T" in ever...

Example 2.40 : $(p \wedge q \wedge r) \vee (p \wedge q \wedge \sim r) \vee (\sim p \wedge q \wedge r) \vee (\sim p \wedge q \wedge \sim r)$ is a PDNF of the simple propositions p, q, r.

Similarly, a compound proposition consisting of conjunctions of maxterms in the variables only and equivalent to a given formula is known as its principal conjunctive normal form (PCNF). PCNF is also called 'product of sum canonical form'.

Example 2.41 : $(p \vee q \vee r) \wedge (p \vee q \vee \sim r) \wedge (\sim p \vee q \vee r) \wedge (\sim p \vee q \vee \sim r)$ is a PCNF of the simple propositions p, q, r.

Note

If the PDNF of a formula 'f' is known, the PDNF of '$\sim f$' will consist of the disjunctions of the remaining minterms which are not included in the PDNF of 'f'.

To obtain the PCNF of 'f', we use the fact that $f \equiv \sim(\sim f)$ and apply De Morgan's laws to the PDNF of '$\sim f$' repeatedly.

2.20 Construction of PDNF and PCNF using truth table

If 'f' is a compound proposition, then we can transform 'f' to PDNF as follows:

- **Step-1.** Build the truth table of the formula 'f'.
- **Step-2.** For each row that gives "T" in the last column of the truth table, generate a conjunction of literals (minterms) corresponding to each row.
- **Step 3.** Take the disjunction of all minterms generated in step 2.

Example 2.42 : Let us consider the truth table for $f \equiv p \leftrightarrow q$

Table 2.27 : $f \equiv p \leftrightarrow q$

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

In the last column of the above truth table, "T" corresponds to the first and last row. The conjunctions of **literals** (minterms) corresponding to first and last row are $(p \wedge q)$ and $(\sim p \wedge \sim q)$. Now the disjunction of the two minterms $(p \wedge q)$ and $(\sim p \wedge \sim q)$ is $(p \wedge q) \vee (\sim p \wedge \sim q)$, which is the required PDNF

If 'f' is a compound proposition, then we can transform 'f' to PCNF as follows:

- **Step 1.** Build the truth table of the formula 'f'.
- **Step 2.** Consider each row that gives "F" in the last column of the truth table. Take the simple propositions for "F" in every row and take negation of the propositions for "T" in every row corresponding to "F" in the last column.

Then generate disjunctions of the literals (maxterms) corresponding to each row.

◆ **Step 3.** Take the conjunction of all maxterms generated in step 2.

Example 2.43 : Let us consider the truth table for $f \equiv p \leftrightarrow q$

Table 2.28 : $f \equiv p \leftrightarrow q$

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

In the last column of the above truth table, "F" corresponds to the second and third row. Now, for the second row, we take $\sim p$ and q corresponding to "T" and "F" respectively. Then we generate the disjunction of $\sim p$ and q i.e., $(\sim p \vee q)$. Now for the third row, we take p and $\sim q$ corresponding to "F" and "T" respectively. Then we generate the disjunction of p and $\sim q$ i.e., $(p \vee \sim q)$. The two maxterms are $(\sim p \vee q)$ and $(p \vee \sim q)$

Now the conjunction of the two maxterms is

$(\sim p \vee q) \wedge (p \vee \sim q)$, which is the required PCNF

Note

The term "literal" means either a variable or a negated variable.

2.21 Boolean Algebra

Boolean algebra is a mathematical paragon based on the set theory and propositional logic. For instance, if one (1) exhibits a sentence is true then zero (0) depicts that this sentence is false, and so we may employ a truth table to display how the truth or falsity of any statement hangs on the truth or falsity of its component statements. The ideas of Boolean algebra are applicable to design and analyze digital switching circuit like personal computers, calculators and many more.

The theory of Boolean algebra was initially suggested by a British mathematician George Boole, in the year 1947 to unravel logical propositions and resolve logic problems. Claude Shannon, consecutive, in his paper "A Symbolic Analysis of Relay and Switching Circuits" remarked on the applications of Boolean algebra in the inspection and design of logical circuits. It is, consequently, employed in designing of computers based on binary numbers 1 and 0. Numbers 1 and 0 accord with two possible states of an electric circuit - current flowing or switch on or off.

There are three fundamental operations $\wedge$ (and), $\vee$ (or) and – (negation) in the event of logical statements, and $\cap$ (intersection), $\cup$ (union), and $(^c)$ (complement)

in set theory. But regarding Boole by AND, OR and NOT, respectiv logical operations. All other ope basic operations. In this chapte (·) and prime (/) shall be used

The motive of the present t algebra of sets, logic and binary electronic digital computers. Algebra.

A non-empty set B on whi multiplication (·) and a unitar said to form a Boolean Algebra are satisfied

(a) Commutative : (i) $a + b$

(b) Each binary operation d

(i) $a + b \cdot c = (a + b)$

(ii) $a \cdot (b + c) = a \cdot b +$

(c) There exist two distinct with respect to additio

(i) $a + 0 = a, \forall a \in$

(d) For each $a \in$ B, there e

(i) $a + a' = 1, \forall a \in$

Note

In the above four postulat be derived from another p

Example 2.44 : Let us co P(X) be the power set o So power set P(X) = {{a Let us define A + B = A

$A \cdot B = A \cap$

Let A′ be the complem Now,

(a) We have

(i) $A + B = A \cup B =$

(ii) $A \cdot B = A \cap B = B$

Thus, the binary commutative.

in set theory. But regarding Boolean algebra all these operations are personalised by AND, OR and NOT, respectively. These fundamental operations are known as logical operations. All other operations may be demonstrated in terms of these basic operations. In this chapter, usual mathematical notations like plus (+), dot (·) and prime (/) shall be used for all above said operations, respectively.

The motive of the present topic is to conclude the fundamental ideas of the algebra of sets, logic and binary arithmetic, and register it to the development of electronic digital computers. Now we shall give a formal definition of Boolean Algebra.

A non-empty set B on which two binary operation namely addition (+) and multiplication (·) and a unitary operation complementation (') are defined, is said to form a Boolean Algebra if the following postulates (due to E.V. Huntington) are satisfied

(a) Commutative : (i) $a + b = b + a$, $\forall a, b \in B$ (ii) $a \cdot b = b \cdot a$, $\forall a, b \in B$

(b) Each binary operation distributes over other :

(i) $a + b \cdot c = (a + b) \cdot (a + c)$ $\forall a, b, c \in B$

(ii) $a \cdot (b + c) = a \cdot b + a \cdot c$, $\forall a, b, c \in B$

(c) There exist two distinct elements O and I in B which are identity elements with respect to addition and multiplication respectively.

(i) $a + O = a$, $\forall a \in B$ (ii) $a \cdot I = a$, $\forall a \in B$

(d) For each $a \in B$, there exists $a' \in B$, such that

(i) $a + a' = I$, $\forall a \in B$ (ii) $a \cdot a' = O$, $\forall a \in B$

Note

In the above four postulates, each postulate has two parts and every part can be derived from another part by using 'Principal of Duality'.

Example 2.44 : Let us consider a set consisting of two elements, $X = \{a, b\}$ and P(X) be the power set of X.

So power set $P(X) = \{\{a\}, \{b\}, \phi, \{a, b\}\}$

Let us define $A + B = A \cup B$, $\forall A, B \in P(X)$

$A \cdot B = A \cap B$, $\forall A, B \in P(X)$

Let A' be the complement of the subset A in X.

Now,

(a) We have

(i) $A + B = A \cup B = B \cup A = B + A$, $\forall A, B \in P(X)$

(ii) $A \cdot B = A \cap B = B \cap A = B \cdot A$, $\forall A, B \in P(X)$

Thus, the binary operation addition (+) and multiplication (·) are commutative.

(b) (i) Now, $A + B \cdot C = A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
$= (A + B) \cdot (A + C)$, $\forall A, B, C \in P(X)$

(ii) $A \cdot (B + C) = A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
$= (A \cdot B) + (A \cdot C)$, $\forall A, B, C \in P(X)$

Thus, each binary operation distributes over other.

(c) Here, (i) $A + \phi = A \cup \phi = A$, $\forall A \in P(X)$

(ii) $A \cdot X = A \cap X = A$, $\forall A \in P(X)$

So, ϕ and X are identity element with respect to addition multiplication $(\cdot)$ respectively.

(d) We have (i) $A + A' = A \cup A' = X$, $\forall a \in B$

(ii) $A \cdot A' = A \cap A' = \phi$, $\forall a \in B$

Thus, P(X) forms a Boolean algebra with respect to two operations addition (+) and multiplication $(\cdot)$ and a unitary op complementation $(')$.

Notes

1. One and all variable in a Boolean algebra has either of two values tr or false (F) which correlates to 1 and 0, respectively.
2. Boolean algebra doesn't have operations for subtraction and division.
3. The distributive law : $a + b \cdot c = (a + b) \cdot (a + c)$ doesn't hold good in ord algebra.

2.22 Duality Principal

The duality of an affirmation in a Boolean algebra B is the affirmation procured by swapping operations (+) and $(\cdot)$ along with swapping the ide 0 and 1 in the primary affirmation. This property of swapping composition identity elements in Boolean algebra is called *Duality principal*. This prin establishes that if a theorem is validated and depends on the axioms of Bo algebra, then a dual theorem procured by swapping (+) with $(\cdot)$ and 0 w also detains. For instant, if $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ then the dual o $a + (b \cdot c) = (a + b) \cdot (a + c)$, similarly, the dual of $(a \cdot 0) + (1 \cdot a) = a$ bec $(a + 1) \cdot (0 + a) = a$.

2.22.1 Some basic properties of Boolean algebra

In simplification of the Boolean expression, the laws and rules of the Bo algebra play an important role. A collection of rules or laws of Boolean Al expressions have been developed to help overcome the number of logic required to perform a paticular logic operation resulting in a list of functi theorems commonly known as Boolean Algebra laws.Now, we discuss some of Boolean algebra and give the proof of these laws.

1 Idempotent law : This la

◆ **Proof :** (i) We may write

RHS = $a = a + 0$ [Identity
$= a + (a \cdot a')$ [Compl
$= (a + a) \cdot (a + a')$ [Di
$= (a + a) \cdot 1 = (a + a)$

(ii) RHS = $a = a \cdot 1$ [Identit
$= a \cdot (a + a')$ [Com
$= (a \cdot a) + (a \cdot a')$ [D
$= (a \cdot a) + 0 = (a \cdot a$

2 Identity law : We write
where, '0' is the identi
element with respect t

◆ **Proof :** (i) Since we ha

RHS = $1 = a + a'$ [Com
$= a + (a' \cdot 1)$
$= (a + a') \cdot (a + 1)$
$= 1 \cdot (a + 1) \cdot (a + 1)$

(ii) Here

LHS = $(a \cdot 0) = 0 + (a$
$= a \cdot a' + a \cdot 0$ [Sin
$= a \cdot (a' + 0)$ [Dis
$= a \cdot a' = 0 = RHS$

3 Absorption law : We
(i) $a + a \cdot b = a$, $\forall a, b \in$

◆ **Proof :** We have

LHS = $a + (a \cdot b) = a \cdot 1$
$= a \cdot (1 + b)$ [Dis
$= a \cdot 1 = a$ [Identi

(ii) Here

LHS = $a \cdot (a + b) = a$
$= a + (a \cdot b)$ [Id
$= a \cdot (1 + b) = a$

4 De Morgan's law : We
(i) $(a + b)' = a' \cdot b'$, $\forall a$

◆ **Proof :** We know that
product is 0.

❶ **Idempotent law :** This law is (i) $a + a = a, \forall a \in B$ (ii) $a \cdot a = a, \forall a \in B$

◆ **Proof :** (i) We may write

RHS = $a = a + 0$ [Identity]

$= a + (a \cdot a')$ [Complement law]

$= (a + a) \cdot (a + a')$ [Distributive law]

$= (a + a) \cdot 1 = (a + a) =$ LHS [Complement law]

(ii) RHS = $a = a \cdot 1$ [Identity]

$= a \cdot (a + a')$ [Complement law]

$= (a \cdot a) + (a \cdot a')$ [Distributive law]

$= (a \cdot a) + 0 = (a \cdot a) =$ LHS [Complement law]

❷ **Identity law :** We write the law (i) $a + 1 = 1, \forall a \in B$ (ii) $a \cdot 0 = 0, \forall a \in B$ where, '0' is the identity element with respect to + and '1' be the identity element with respect to '·'

◆ **Proof :** (i) Since we have

RHS = $1 = a + a'$ [Complement law]

$= a + (a' \cdot 1)$

$= (a + a') \cdot (a + 1)$ [Distributive law]

$= 1 \cdot (a + 1) \cdot (a + 1) =$ LHS [Complement law]

(ii) Here

LHS = $(a \cdot 0) = 0 + (a \cdot 0)$ [Complement law]

$= a \cdot a' + a \cdot 0$ [Since $a \cdot a' = 0$]

$= a \cdot (a' + 0)$ [Distributive law]

$= a \cdot a' = 0 =$ RHS [Complement law]

❸ **Absorption law :** We have the law as

(i) $a + a \cdot b = a, \forall a, b \in B$ (ii) $a \cdot (a + b) = a, \forall a, b \in B$

◆ **Proof :** We have

LHS = $a + (a \cdot b) = a \cdot 1 + (a \cdot b)$ [Identity]

$= a \cdot (1 + b)$ [Distributive law]

$= a \cdot 1 = a$ [Identity law]

(ii) Here

LHS = $a \cdot (a + b) = a \cdot a + a \cdot b$ [Distributive law]

$= a + (a \cdot b)$ [Idempotent law]

$= a \cdot (1 + b) = a \cdot 1 = a =$ RHS [Distributive law]

❹ **De Morgan's law :** We write the law as

(i) $(a + b)' = a' \cdot b', \ \forall a, b \in B$ (ii) $(a \cdot b)' = a' + b', \forall a, b \in B$

◆ **Proof :** We know that, the sum of a variable and its complement is 1 and their product is 0.

Let, $A = a + b$, then $A' = (a + b)'$

In order to establish the first part of the theorem, we have to prove that $a + b + a'\cdot b'$ should be equal to 1 and $(a + b)\cdot(a'\cdot b')$ should be equal to 0

Now, $a + b + a'b' = [(a + b) + a'][(a + b) + b']$ [by using Distributive law]

$= (a + a' + b)\cdot(a + b + b') = (1 + b)\cdot(a + 1) = 1\cdot 1 = 1$

Again, $a + b\cdot a'b' = (a\cdot a'\cdot b') + (b\cdot a'\cdot b')$ [by using Distributive law]

$= (0\cdot b') + (0\cdot a') = 0 + 0 = 0$

Hence, $(a + b)' = a'\cdot b'$, $\forall a, b \in B$

Now, by using Principal of Duality on the obtained result, we can find the second part of the theorem. This is left as an exercise for the readers.

Now, we state some other important laws on Boolean algebra :

5. **Associative law :** The law states

(i) $a + (b + c) = (a + b) + c$, $\forall a, b, c \in B$ (ii) $a\cdot(b\cdot c) = (a\cdot b)\cdot c$, $\forall a, b, c \in B$

6. **Cancellation law :** The law states

If $x + z = y + z$ and $x\cdot z = y\cdot z$, then $x = y$, where, $x, y, z \in B$

7. **Involution law :** The law states

(i) $(a')' = a$, (ii) $(1)' = 0$ (iii) $(0)' = 1$

2.22.2 Boolean Function

To simplify the definition of Boolean function, we shall be using [illegible] for operations from chapters on Set Theory and Mathematical Logic. [illegible] function is very helpful in understanding circuits in switching theory. Let [illegible] a formal defibition of Boolean Function.

Let $(B, +, \cdot, ')$ be a Boolean algebra. Let $x_1, x_2, ..., x_n$ be variables (stat[illegible] in B which can assume values 0 and 1. The equation generated by the [illegible] $x_1, x_2, ..., x_n$ is valid combination of x_i's with the operations defined on A.

If $(x_1, x_2, ..., x_n)$ is a Boolean expression, then the function $f(x_1, x_2, ...$ generated by n variables is defined as: $f : A_n \to A$.

The Boolean expression represents a compound statement [illegible] combinations of 0s and 1s for variables x_i's in the expression.

The values 0 and 1 for a statement in the function $f(x_1, x_2, ..., x_n)$ [illegible] that the statement is false and true, respectively and hence truth or [illegible] $f(x_1, x_2, ..., x_n)$ depends on the truth or falsity of its component statements.

Boolean functions may be used to represent output requirements of [illegible] for any possible input values as shown in Fig 2.1. Each variable x_i [illegible] indicator voltages, *i.e.*, one for 0 and other for 1. Thus for all such [illegible] function f indicates output response. Such requirements are used while [illegible] combinatorial and sequential computer circuits.

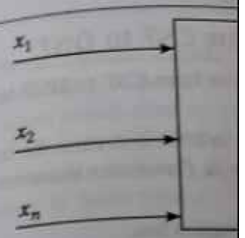


Fig.

2.22.3 Disjunctive [illegible] normal form (CNF)

Let us consider a Boole[illegible] $f(x, y) = xy' + yx' + x'y'$. This is [illegible]

Now, we consider another Bo[illegible]

$\sigma(x, y) =$

Since σ contains four (= [illegible] variables. Again, if we consi[illegible] $g(x, y, z) = xyz + x'y'z' + x'yz +$ [illegible]

So complete DNF for three [illegible]

For example, $h(x, y, z) = xy$ [illegible] a complete DNF expression.

Now, we consider another [illegible] $u(x, y)$ is in Conjunctive norm[illegible] four $(= 2^2)$ factors, then $u(x,$ [illegible] of two variables $m(x, y) = (x +$ [illegible] in complete CNF form.

2.22.4 Working pro[illegible]

The working procedure fo[illegible] as follows

- **Step-1 :** Suppose a Boo[illegible] compute f' by using co[illegible] of complete D.N.F whi[illegible]
- **Step-2 :** $f = (f')'$ can be [illegible]

Example 2.45 : We show [illegible] to CNF. Here σ' can be [illegible] contains those terms, wh[illegible]

So, $\sigma' = x'y'z + x'yz' + x'$[illegible]

Now, $\sigma = (\sigma')' = (x'y'z +$ [illegible]

$= (x'y'z)'\cdot(x'yz')'\cdot($[illegible]

$= (x + y + z')\cdot(x +$ [illegible]

which is in CNF.

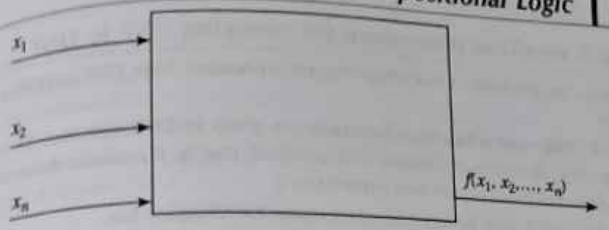


Fig. 2.1 Input Output Diagram

2.22.3 Disjunctive normal form (DNF) and Conjunctive normal form (CNF)

Let us consider a Boolean function f of two variables x, y. Suppose $f(x, y) = xy' + yx' + x'y'$. This is in Disjunctive normal form (DNF) for two variables. Now, we consider another Boolean function of two variables,

$$\sigma(x, y) = xy' + yx' + x'y' + xy.$$

Since σ contains four $(= 2^2)$ terms, so $\sigma(x, y)$ is in complete DNF for two variables. Again, if we consider another Boolean function of three variables $g(x, y, z) = xyz + x'y'z' + x'yz + xy'z$. So, $g(x, y, z)$ is in DNF for three variables.

So complete DNF for three variables will contain eight $(= 2^3)$ terms.

For example, $h(x, y, z) = xyz + x'y'z' + x'yz + xy'z + xyz' + x'y'z + xy'z' + x'yz'$ is a complete DNF expression.

Now, we consider another Boolean function $w(x, y) = (x + y)(x' + y')(x' + y)$. So $w(x, y)$ is in Conjunctive normal form (CNF) for two variables. If $w(x, y)$ contains four $(= 2^2)$ factors, then $w(x, y)$ is in complete CNF form. The Boolean function of two variables $m(x, y) = (x + y)(x' + y')(x' + y)(x + y')$ is an example of a function in complete CNF form.

2.22.4 Working procedure for reducing DNF to CNF

The working procedure for reducing an expression from DNF to CNF is given as follows

- **Step-1 :** Suppose a Boolean function f is given in DNF, and then we have to compute f' by using complete DNF method, that is, f' contains those terms of complete D.N.F. which are not present in f.
- **Step-2 :** $f = (f')'$ can be computed by using De Morgan's law.

Example 2.45 : We show the procedure to reduce $\sigma = xy'z + xy'z' + xyz + x'y'z'$ to CNF. Here σ' can be computed by complete DNF method. That is σ' contains those terms, which are present in complete DNF but not in σ.

So, $\sigma' = x'y'z + x'yz' + x'yz + xyz'$

Now, $\sigma = (\sigma')' = (x'y'z + x'yz' + x'yz + xyz')'$

$= (x'y'z)' \cdot (x'yz')' \cdot (x'yz)' \cdot (xyz')'$ [by De-Morgan's law]

$= (x + y + z') \cdot (x + y' + z) \cdot (x + y' + z') \cdot (x' + y' + z)$ [by De-Morgan's Law]

which is in CNF.

2.22.5 Working procedure for reducing CNF to DNF

The working procedure for reducing an expression from CNF to DNF as follows

- **Step-1** : Suppose a Boolean function f is given in CNF, and then we compute f' by using complete CNF method, that is, f' contains those complete CNF which are not present in f.
- **Step-2** : $f = (f')'$ can be evaluated by using De Morgan's law.

Example 2.46 : We show the procedure to reduce $\sigma = (x + y)(x' + y)$ D.N.F. Here σ' can be computed by complete CNF method. That is σ' those factors, which are present in complete CNF but not in σ.

Thus $\sigma' = (x + y')$

Now, $\sigma = (\sigma')' = (x + y')' = x'y$ [by using De Morgan's law]

2.22.6 Some applications of Boolean Algebra

One of the practical applications of Boolean algebra is to facilitate the design of certain electrical network (or circuit). An electrical switch has two states: closed and open, i.e one allowing the current to pass (closed state) one not allowing the current to pass (open state). A common household switch is an example of two-state switch.

The standard notation used for Boolean algebra operations in the design of switching circuits are (+) for $\vee$ or $\cup$, (·) for or $\wedge$ or $\cap$ and (-) for (complement)

In Boolean algebra digits 0 and 1 do not represent actual numbers terms that are used with 0 and 1 in the digital logic field are given in the table :

Table 2.29 : Digital logic

Logic 0	Logic 1
False	True
Off	On
Open	Closed
No	Yes
F	Y

We shall now descripe Open and Closed Circuit. Consider electrical as shown in Fig. 2.2(a) and 2.2 (b).

Fig 2.2(a) : Open Circuit

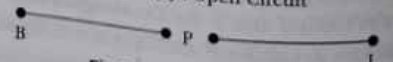


Fig 2.2(b) : Closed Circuit

In these diagrams p represents a switch an electric lamp. A switch is said to be op i.e. an open switch does not allow the flow the flow of current. The value of a closed of an open switch when L does not light are shown in Table 2.30.

Table 2.30 : St

State of Switch p
0 (open)
1 (closed)

Boolean Multiplication (Switches in

The logical multiplication in Bool operation $p \wedge q$ or pq circuit. The logic is expressed as : $p \cdot q = r$ and is read as q and r may have only values 0 or 1. A is as follows

If in an electric circuit, two switches 2.3, then the switches are said to be in

Fig 2.3

In general, four different states ma

(i) p open, q open
(ii) p open, q closed
(iii) p closed, q closed
(iv) p closed, q open

2.23 Switching Circuit

A switch has two states, open or clo connected switches, with two connec system, switches may be connected series connection or through the us switch x the corresponding complem describe switching systems formally closed states by 0 (F) and 1(T) respec we use $x + y (x \vee y)$ and $x \cdot y (x \wedge y)$ to d respectively. The symbols "·" and "+ in arithmetic, although there exist se notations, we can represent the state in series.

In these diagrams p represents a switch, B represents a battery and L represents an electric lamp. A switch is said to be open if L does not light otherwise closed. I.e. an open switch does not allow the flow of current while a closed switch allows the flow of current. The value of a closed switch when L lights is 1(one) and value of an open switch when L does not light is 0 (zero). The status of switch and lamp are shown in Table 2.30.

Table 2.30 : States of a Switch

State of Switch p	State of Lamp L
0 (open)	0 (off)
1 (closed)	1 (ON)

Boolean Multiplication (Switches in Series) is described below :

The logical multiplication in Boolean Algebra is also referred to as AND operation $p \wedge q$ or pq circuit. The logical multiplication of two variables p and q is expressed as : $p \cdot q = r$ and is read as: p AND q equals r. Each of the variables p, q and r may have only values 0 or 1. An application of multiplication operation is as follows

If in an electric circuit, two switches p and q are connected as shown in Figure 2.3, then the switches are said to be in series.

Fig 2.3 : Series Connection

In general, four different states may exist in these type of circuits, namely

(i) p open, q open
(ii) p open, q closed
(iii) p closed, q closed
(iv) p closed, q open

2.23 Switching Circuit

A switch has two states, open or closed and a switching system is a collection of connected switches, with two connection points available outside. In a switching system, switches may be connected with one another either in parallel or in series connection or through the use of complementary switches. For a given switch x the corresponding complementary switch, is denoted by x'. In order to describe switching systems formally and mathematically, we denote open and closed states by 0 (F) and 1(T) respectively. Further for any two switches x and y, we use $x + y (x \vee y)$ and $x \cdot y (x \wedge y)$ to denote their parallel and series connections respectively. The symbols "·" and "+" here should not be confused with those in arithmetic, although there exist some similarities. With the above introduced notations, we can represent the state of or the effect of switches in parallel, and in series

- **Parallel connection circuit :** A circuit consisting of two switches connected in parallel, is denoted by $x + y (x \vee y)$

Fig 2.4

i.e, $x + y$ is closed if and only if at least one of x, y is closed.

So,

Table 2.31(a)

x	y	$x + y$
0	0	0
0	1	1
1	0	1
1	1	1

- **Series connection circuit :** Two switches x and y connected in se denoted by $x \cdot y (x \wedge y)$.

Fig 2.5

So,

Table 2.31(b)

x	y	$x \cdot y$
0	0	0
0	1	0
1	0	0
1	1	1

i.e, $x \cdot y$ is closed if and only if both x and y are closed.

Example 2.47 : We can write the switching circuit corresponding switching function

$f = [xy'z + y(z + x')](x' + y') + y'$

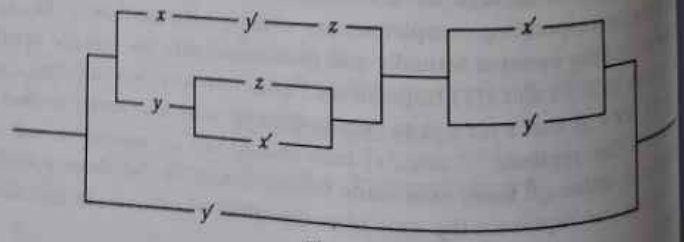
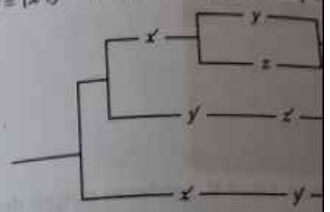
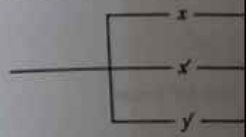


Fig 2.6

Example 2.48 : The switching circuit
$f = [x'(y + z) + y'z'](x' + y + z') + x'y($



Example 2.49 : We can simplify t



as follows :

$f = xyz + x'z + y'z$

$= xyz + (x' + y')z$, Since each bin

$= (xy)z + (xy)'z$, By De-Morgan's

$= [xy + (xy)']z$

$= 1 \cdot z$, for each x, x', we have $x +$

$= z$

This is the simplified equivalen

2.24 Introduction to l

A logic gate is an electronic cir these decisions, the most commo NAND gates. The NOR and NAND we describe logical symbols and t two inputs to give preliminary id

- **OR gate :** An OR gate receive A + B, as shown in logic table

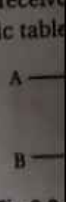


Fig 2.9

Example 2.48 : The switching circuit corresponding to the switching function $f = (x'(y + z) + y'z')(x' + y + z') + x'y'(x + y + z')$

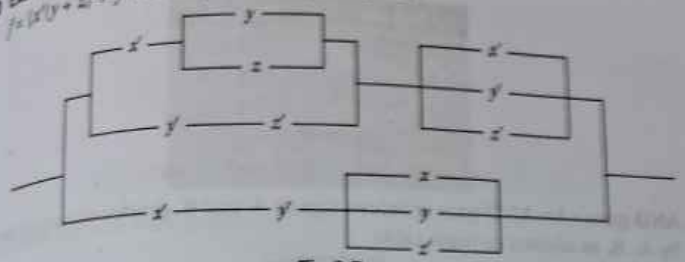


Fig 2.7

Example 2.49 : We can simplify the following switching circuit.

Fig 2.8

as follows :

$f = xyz + x'z + y'z$

$= xyz + (x' + y')z$, Since each binary operation distributes over other.

$= (xy)z + (xy)'z$, By De-Morgan's law

$= [xy + (xy)']z$

$= 1 \cdot z$, for each x, x', we have $x + x' = 1$

$= z$

This is the simplified equivalent circuit,

2.24 Introduction to Logic Gates

A logic gate is an electronic circuit which makes logical decisions. To arrive at these decisions, the most common logic gates used are OR, AND, NOT, NOR and NAND gates. The NOR and NAND gates are called Universal gates. In this section, we describe logical symbols and truth tables for the above mentioned gates with two inputs to give preliminary ideas about gates.

- **OR gate :** An OR gate receives inputs A and B, produces output denoted by A + B, as shown in logic table.

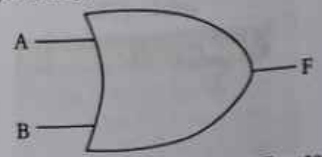


Fig 2.9. Logical symbols OR gate

The truth table of OR gate is

Table 2.32(a)

Inputs		Output
A	B	F = A + B
0	0	0
0	1	1
1	0	1
1	1	1

- **AND gate :** An AND gate receives inputs A and B, produces output denoted by A · B, as shown in logic table.

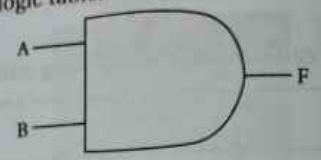


Fig 2.10. Logical symbols AND gate

The truth table of AND gate is

Table 2.32 (b)

Inputs		Output
A	B	F = A·B
0	0	0
0	1	0
1	0	0
1	1	1

- **NOT gate:** A NOT gate receives input A, produces output F denoted by shown in logic table.

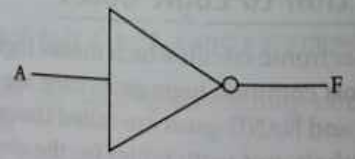


Fig 2.11. Logical symbols NOT gate

The truth table of NOT gate is

Table 2.33

Inputs	Output
A	F = A′
0	1
1	0

- **NOR gate :** The NOR gate is a digital logic gate that implements logical NOR, it behaves according to the truth table given below.

Fig 2.12. Logical symbols NOR gate

The truth table of NOR gate is

Table 2.34

Inputs		Output
A	B	$F = \overline{A+B}$
0	0	1
0	1	0
1	0	0
1	1	0

- **NAND gate :** A NAND gate is a NOT-AND logic gate that produces an output. It is a combination of two basic logic gates, the AND gate and the NOT gate connected in series. So, NAND gate and NOR gate parallel combination can produce an inverter, an OR gate or an AND gate. An NAND gate receives inputs A and B, produces output denoted $\overline{AB}$, as shown in logic table.

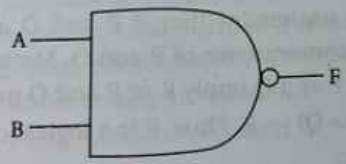


Fig 2.13. Logical symbols NAND gate

Truth table of NAND gate is

Table 2.35

Inputs		Output
A	B	$F = \overline{A \cdot B}$
0	0	1
0	1	1
1	0	1
1	1	0

2.25 Theorem

An axiom is a statement with true value T, which is to serve as a
An axiom serves as a initial point from which other statements can be
derived. A statement formula β is said to be provable if there exists a pro
formula that is provable is called a *theorem*. It is a statement that has been
to be true based on some axioms.

A theorem of a formal theory T_f is a *WFF* α of T_f such that α is the
of some proof in T_f. Here, a proof is called a proof of α in T_f. A theory is
there is an effective procedure is said to be *decidable*, otherwise, the
called *undecidable*.

2.26 Logical Consequence

Modal logic is the study of logic that involves the use of the expr
'necessarily' and 'possibly'. However, the term 'modal logic' is used more
to cover a family of logics with similar rules and a variety of different sy
modal is that which is used to qualify **the truth of a judgement.**

The set of *WFF* Σ is a system of modal logic iff it contains all propo
tautologies (PL) and is closed under modus ponens (MP) and uniform subs
(US). The symbol $\vdash_\Sigma Q$ means that Q is a theorem of Σ that is $\vdash_\Sigma Q$ iff Q

Let us consider the following statements :

P : Ram is taller than Shyam

Q : Shyam is taller than Raja

R : Ram is taller than Raja

We can justify our stastement that if P and Q are true then R is
this case R is a logical consequence of P and Q. Mathematically, we can
as P, Q $\vdash$ R (this means P and Q imply R or P and Q prove R).

We justify this as $(P \wedge Q) \rightarrow R$. Thus, R is a logical consequence of $P \wedge$

2.26.1 Logical Deducibility and logical consistency

A WFF 'Q' is deducible from a set of formulas Γ in a logic Σ — written
— iff Σ contains a theorem of the form $(P_1 \wedge \cdots \wedge P_n) \rightarrow Q$ where the co
$P_1 \wedge \cdots \wedge P_n$ are formulas in Γ. It is convenient to extend the notation. A
set of formulas, $\Gamma \vdash_\Sigma \Gamma'$ means that $\Gamma \vdash_\Sigma Q$ for every Q in Γ'. A set of formu
inconsistent in Σ (Σ-inconsistent) just in case $\perp$ is Σ-deducible from Γ. A
formulas Σ-consistent when it is not Σ-inconsistent.

- **Deducibility** : We write mathematically $\Gamma \vdash_{,\Sigma} Q$ iff there are formulas
  $P_1 \wedge \ldots \wedge P_n \in \Gamma$ ($n \geq 0$) such that $\vdash_\Sigma (P_1 \wedge \ldots \wedge P_n) \rightarrow Q$. For Γ' a set of fo
  $\Gamma \vdash_\Sigma \Gamma'$ means that $\Gamma \vdash_\Sigma Q$ for every Q in Γ' .

- **Consistency** : We write mathem
  Σ-inconsistent iff $\Gamma \vdash_\Sigma \perp$.

Some Theorems of consistency

- **Theorem 1 :** $\vdash_\Sigma Q$ iff $\phi \vdash_\Sigma Q$
- **Theorem 2 :** $\vdash_\Sigma Q$ iff for eve
- **Theorem 3 :** If $\Gamma \vdash_{PL} Q$, the
- **Theorem 4 :** If $Q \in \Gamma$ then
  formulas, $\Gamma \vdash_\Sigma \Gamma$.)
- **Theorem 5 :** If $\Gamma \vdash_\Sigma R$ and
  set of formulas : if $\Gamma \vdash_\Sigma \Gamma'$
- **Theorem 6 :** If $\Gamma \vdash_\Sigma Q$ and
- **Theorem 7 :** $\Gamma \vdash_\Sigma Q$ iff the
- **Theorem 8 :** $\Gamma \vdash_\Sigma Q \rightarrow R$
- **Theorem 9 :** Γ is Σ-consi
- **Theorem 10 :** Γ is Σ-cons
  $\Gamma \vdash \Sigma -Q$.
- **Theorem 11 :** If Γ is Σ-c
- **Theorem 12 :** If Γ is Σ-c
- **Theorem 13 :** Γ is Σ-con
- **Theorem14 :** $\Gamma \vdash_\Sigma Q$ iff
- **Proof :** The proofs of t

2.26.2 Maximal cons

A set of sentences is maxi
short) just in case it is Σ-consi
In other words, a set is Σ-ma
formulas as it can without be

So, a set of formulas Γ is
(i) Γ is Σ-consistent, and
(ii) for every formula Q,

We note that the clause s
a formula not already in Γ y

Here are some theorems

- **Theorem 1 :** $Q \in \Gamma$
- **Theorem 2 :** $\Sigma \subseteq \Gamma$.
- **Theorem 3 :** $-Q \in \Gamma$
- **Theorem 4 :** $Q \wedge R$
- **Theorem 5 :** $Q \vee R$
- **Theorem 6 :** $Q \rightarrow R$

- **Consistency** : We write mathematically Γ is Σ-consistent iff not $\Gamma \vdash_\Sigma \bot$. Γ is Σ-inconsistent iff $\Gamma \vdash_\Sigma \bot$.

Some Theorems of consistency and deducibility is as follows :

- **Theorem 1 :** $\vdash_\Sigma Q$ iff $\phi \vdash_\Sigma Q$.
- **Theorem 2 :** $\vdash_\Sigma Q$ iff for every Γ, $\Gamma \vdash_\Sigma Q$.
- **Theorem 3 :** If $\Gamma \vdash_{PL} Q$, then $\Gamma \vdash_\Sigma Q$.
- **Theorem 4 :** If $Q \in \Gamma$ then $\Gamma \vdash_\Sigma Q$. (Or using the $\vdash_\Sigma$ notation for sets of formulas, $\Gamma \vdash_\Sigma \Gamma$.)
- **Theorem 5 :** If $\Gamma \vdash_\Sigma R$ and $\{R\} \vdash_\Sigma Q$, then $\Gamma \vdash_\Sigma Q$. More generally, for Γ' any set of formulas : if $\Gamma \vdash_\Sigma \Gamma'$ and $\Gamma' \vdash_\Sigma Q$, then $\Gamma \vdash_\Sigma Q$.
- **Theorem 6 :** If $\Gamma \vdash_\Sigma Q$ and $\Gamma \subseteq \Gamma'$, then $\Gamma' \vdash_\Sigma Q$.
- **Theorem 7 :** $\Gamma \vdash_\Sigma Q$ iff there is a finite subset Γ_x of Γ such that $\Gamma_x \vdash_\Sigma Q$.
- **Theorem 8 :** $\Gamma \vdash_\Sigma Q \to R$ iff $\Gamma \cup \{Q\} \vdash_\Sigma R$.
- **Theorem 9 :** Γ is Σ-consistent iff there is an Q such that not $\Gamma \vdash \Sigma Q$.
- **Theorem 10 :** Γ is Σ-consistent iff there is no Q such that both $\Gamma \vdash \Sigma Q$ and $\Gamma \vdash \Sigma \neg Q$.
- **Theorem 11 :** If Γ is Σ-consistent, then Γ is PL-consistent.
- **Theorem 12 :** If Γ is Σ-consistent and $\Gamma' \subseteq \Gamma$, then Γ' is Σ-consistent.
- **Theorem 13 :** Γ is Σ-consistent iff every finite subset Γ_x of Γ is Σ-consistent.
- **Theorem14 :** $\Gamma \vdash_\Sigma Q$ iff $\Gamma \cup \{\neg Q\}$ is Σ-inconsistent.
- **Proof :** The proofs of these theorems are beyond the scope of the book.

2.26.2 Maximal consistent sets

A set of sentences is maximal consistent in a system Σ (Σ-maxi-consistent for short) just in case it is Σ-consistent and has only Σ-inconsistent proper extensions. In other words, a set is Σ-maxi-consistent if it is consistent and contains as many formulas as it can without becoming inconsistent.

So, a set of formulas Γ is Σ-maxi-consistent if

(i) Γ is Σ-consistent, and

(ii) for every formula Q, if $\Gamma \cup \{Q\}$ is Σ-consistent, then $Q \in \Gamma$.

We note that the clause says that where Γ is Σ-maxi-consistent, the addition of a formula not already in Γ yields a Σ-inconsistent set of formulas.

Here are some theorems of Σ-maxi-consistent sets.

- **Theorem 1 :** $Q \in \Gamma \Leftrightarrow \Gamma \vdash \Sigma Q$.
- **Theorem 2 :** $\Sigma \subseteq \Gamma$.
- **Theorem 3 :** $\neg Q \in \Gamma \Leftrightarrow Q \notin \Gamma$.
- **Theorem 4 :** $Q \wedge R \in \Gamma \Leftrightarrow Q \in \Gamma$ and $R \in \Gamma$.
- **Theorem 5 :** $Q \vee R \in \Gamma \Leftrightarrow Q \in \Gamma$ or $Q \in \Gamma$.
- **Theorem 6 :** $Q \to R \in \Gamma \Leftrightarrow (Q \in \Gamma \Rightarrow R \in \Gamma)$.

- **Theorem 7 :** $Q \leftrightarrow R \in \Gamma \Leftrightarrow (Q \in \Gamma \Leftrightarrow R \in \Gamma)$.
- **Theorem 8 :** Γ is a Σ-system.
- **Proof :** The proofs of these theorems are beyond the scope of the

2.26.3 Lindenbaum's Lemma

Let Γ be a Σ-consistent set of formulas. Then there exists a Σ-maxi-con set Δ such that $\Gamma \subseteq \Delta$

There is a relationship between deducibility in Σ ($\Gamma \vdash_\Sigma Q$) and Σ- consistent sets. From Lindenbaum's lemma it follows that a formula Q is ded from a set of formulas Γ if and only if Q belongs to every maximal extension And a formula Q is a theorem of Σ(i.e. $\vdash_\Sigma Q$) if and only if Q is a member of Σ-maxi-consistent set.

Here are some related theorems

- **Theorem 1 :** $\Gamma \vdash \Sigma A$ iff $A \in \Delta$ for every Σ-maxi-consistent Δ such that Γ
- **Theorem 2 :** $\vdash \Sigma A$ iff $A \in \Delta$ for every Σ-maxi-consistent Δ.
- **Proof :** The proofs of these theorems are beyond the scope of the bo

2.27 Completeness and Soundness

There are two tracks which are used in propositional logic. One is sema methodology and anoher one is syntactic technique. In semantic meth the propositional forms are assigned with the truth values T or F and then can determine the tuth value of any proposition associated with the defi propositional form. In syntactic method, any proof or formula is deduced f the axioms or different inference rules. The question arises whether there is relation between the two methodologies.

Now, we would like to prove a propositional form α with using the formula It is provable, we can write mathematically $p \vdash \alpha$. The proof will be valid if is a truth assignment as $f(p)$= T and $f(\alpha)$ = T.

Let, $P_1, P_2, ..., P_n$ be a group of compound propositions that yields conclusion Q. Symbolically we write as $P_1, P_2, ..., P_n \vdash Q$. When we proved $P_1, P_2, ..., P_n \vdash Q$, the idea was that eventually we should be able to show that all of $P_1, P_2, ..., P_n$ were true, then Q would also be true. Now we have a way describing a circumstance (a valuation) under which we can evaluate the value of formulas. This suggests that what we want to show is that any valuation which makes, $P_1, P_2, ..., P_n$ true should make Q true. If this is in fact the case we write $P_1, P_2, ..., P_n \models Q$. Here the symbol $\models$ is read as "models". We need establish the fact about the relationship between the two relations $\models$ and $\vdash$.

We define $P_1, P_2, ..., P_n \models Q$ to mean that any valuation that makes P_1, P_2 P_n true also makes Q true. $\models$ is also called the *semantic entailment relation*.

The property that if $P_1, P_2, ..., P_n \vdash Q$ then $P_1, P_2, ..., P_n \models Q$ is called **soundness**. The property that if $P_1, P_2, ..., P_n \models Q$, then $P_1, P_2, ..., P_n \vdash Q$ is called **completeness**.

Soundness states that any valid between the formulas involved, u given a relationship between the v of validity of the corresponding se (the number of formulas on the le formula that is a theorem is true u formula that is true under all valu two properties for our system of Soundness asserts that a va A sequent is shown valid by me soundness will proceed by indu lines). A logic is said to be sound

2.28 Theorems on C

- **Theorem 1 :** $P_1, P_2, ..., P_n \vdash$
- **Proof :** We shall therefore as deduction rules and proceed the length k of a proof of the it must be of the form

1 P prem

So the sequent is $P \vdash P$, an For the inductive step, we as of length less than k, and pr a proof of length k (again, only core rules). The kth li have a number of cases de do enough cases to get the deduction.

If the kth line looks like k (then Q must be of the fo appears on line k_2.

The sequent $P_1, P_2, ..., P_n \vdash$ of the proof we started wit $P_1, P_2, ..., P_n \models Q_2$.

What this means is that fo

$\Phi(P_1) = T, ..., \Phi(P_n) = T,$

whenever $\Phi(P_1) = T, ...$

So let $\Phi(P_1) = ... = \Phi(P_n) =$

Therefore, $\Phi(Q_1 \wedge Q_2) =$

Thus, $P_1, P_2, ..., P_n \models Q$, a

Soundness states that any valid sequent says something about the relationship between the formulas involved, under all valuations. Completeness states that given a relationship between the valuations of formulas, there must exist a proof of validity of the corresponding sequent. It helps to think about the case where n (the number of formulas on the left-hand side) is zero. Soundness states that any formula that is a theorem is true under all valuations. Completeness says that any formula that is true under all valuations is a theorem. We are going to prove these two properties for our system of natural deduction and our system of valuations.

Soundness asserts that a valid sequent says something about valuations. A sequent is shown valid by means of a natural deduction proof. Our proof of soundness will proceed by induction on the length of this proof (the number of lines). A logic is said to be sound if every theorem is a tautology.

2.28 Theorems on Completeness and Soundness

Theorem 1 : $P_1, P_2, ..., P_n \vdash Q$, then $P_1, P_2, ..., P_n \models Q$

Proof : We shall therefore assume that that the given proof of 'P' uses only core deduction rules and proceed by strong induction (mathematical induction) on the length k of a proof of the sequent $P_1, P_2, ..., P_n \vdash Q$. If the proof has one line, it must be of the form

1 P premise

So the sequent is $P \vdash P$, and clearly $P \models P$

For the inductive step, we assume that soundness holds for sequents with proofs of length less than k, and prove that it holds for a sequent $P_1, P_2, ..., P_n \vdash Q$ with a proof of length k (again, our attention is restricted to deductions involving only core rules). The kth line of the proof must look like k Q justification We have a number of cases depending on which rule is applied in this line. We'll do enough cases to get the general idea, since we have a lot of rules in natural deduction.

If the kth line looks like $k\ Q \wedge i\ k_1, k_2$

then Q must be of the form $Q_1 \wedge Q_2$, where Q_1 appears on line k_1, and Q_2 appears on line k_2.

The sequent $P_1, P_2, ..., P_n \vdash Q$ has a proof of length at most k_1 (the first k_1 lines of the proof we started with), so $P_1, P_2, ..., P_n \models Q_1$. Similarly,

$P_1, P_2, ..., P_n \models Q_2$.

What this means is that for every interpretation Φ such that

$\Phi(P_1) = T, ... , \Phi(P_n) = T$, we also have $\Phi(Q_1) = T$, and

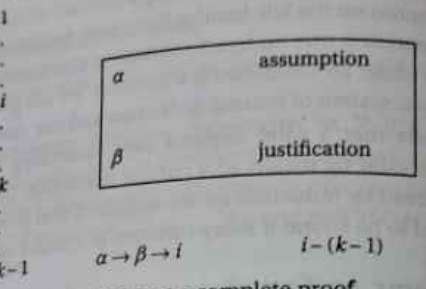
whenever $\Phi(P_1) = T, ..., \Phi(P_n) = T$, we have $\Phi(Q_2) = T$.

So let $\Phi(P_1) = ... = \Phi(P_n) = T$. Then $\Phi(Q_1) = \Phi(Q_2) = T$.

Therefore, $\Phi(Q_1 \wedge Q_2)$ = meaning $(\wedge)(\Phi(Q_1)), \Phi(Q_2))$ = meaning $(\wedge)(T, T) = T$.

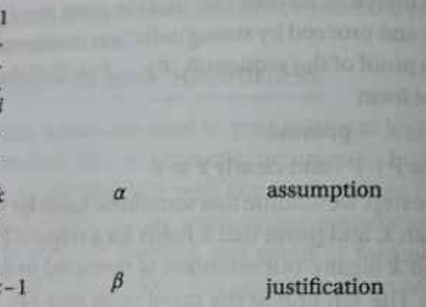
Thus, $P_1, P_2, ..., P_n \models Q$, as required.

This sort of reasoning works well for many other rules, such as $\wedge e_1$ and ... But it gets a little complicated if the last line in our proof closes off a box with $\rightarrow i$. Suppose we have such a proof of $P_1, P_2, ..., P_n \models \alpha \rightarrow \beta$



If we delete the last line, this is not a complete proof.

But we can turn it into one by making the assumption of the open box into a premise.



This is a proof of the sequent $P_1, P_2, ..., P_n, \alpha \vdash \beta$.

Since our proof of $P_1, P_2, ..., P_n, \alpha \vdash \beta$ has $k-1$ lines, we can apply the induction hypothesis to conclude that $P_1, P_2, ..., P_n, \alpha \models \beta$

We need to conclude that $P_1, P_2, ..., P_n \models \alpha \rightarrow \beta$. So consider an interpretation Φ that makes $P_1, P_2, ..., P_n$ true. The only way we could have $\Phi(\alpha \rightarrow \beta) =$ would be if $\Phi(\alpha) = T$ and $\Phi(\beta) = F$ (this is by examination of the truth table that defines meaning($\rightarrow$)).

But we have $P_1, P_2, ..., P_n, \alpha \models \beta$, so if $\Phi(\phi_1) = \cdots = \Phi(\phi_n) = \Phi(\alpha) = T$, then $\Phi(\beta) = T$.

Hence $\Phi(\alpha \rightarrow \beta) = T$, and we have $P_1, P_2, ..., P_n \models \alpha \rightarrow \beta$

This sort of reasoning works for the cases where the last line closes off a proof box. The full proof involves a complete examination of each of the dozen or so rules of natural deduction, but we have already seen the two main ideas. Soundness gives us a method of showing that a sequent $P_1, P_2, ..., P_n \vdash Q$ does not have a proof in natural deduction; we simply have to find a valuation that makes $P_1, P_2, ..., P_n$ true but Q false

It is worth noting that the proof of soundness actually holds for both classical and intuitionistic logic. It simply one that does not employ the rule PBC, so it is also a classical proof, and therefore a consequence of soundness for classicis

Theorem 2 : $P_1, P_2, ..., P_n \models Q$, then P_1

Proof : The proof is divided into three

The first stage shows that if $P_1, P_2, ..., P$

The second stage shows that if $\models \eta$, th

looking like what the first stage produ

The third stage shows that if '$P_1 \rightarrow (P_2$

The first and third stages are the eas

first need to show that if $P_1, P_2, ...,$ P

How can we have $\Phi(P_1 \rightarrow (P_2 \rightarrow (...($

would need $\Phi(P_1) = T$ and $\Phi(P_1 \rightarrow$ (

to unravel the formula, and conclu

$\Phi(Q) = F$. But we know $P_1, P_2, ..., P_n$

$\models$. For the third stage, we must take

from it construct a proof of $P_1, P_2,$

This can be done simply by augmen

with n lines at the beginning intro

at the end, each applying $\rightarrow e$ to p

leaves ψ as the last line. The resul

speaking, both of these stages sh

are so "iterative" or algorithmic i

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do a structural induction on form

as possible.

The second stage is more comp

though it is not technically diffi

η, if $\models \eta$, then η. Suppose η h

different from the one in the fi

construct a proof knowing only

to η. We do this by constructin

and then putting them all toget

of the atoms, which we'll call p

truth table for η.) Relative to Φ

otherwise $\hat{p}_i$ is the formula $\neg p_i$

It is worth noting that the proof of soundness given here for classical deductions actually holds for both classical and intuitionist proofs. An intuitionist proof is simply one that does not employ the $\sim\sim e$ rule; hence any intuitionist proof is also a classical proof, and therefore soundness for intuitionism is an easy consequence of soundness for classicism.

Theorem 2 : $P_1, P_2, \ldots, P_n \models Q$, then $P_1, P_2, \ldots, P_n \vdash Q$

Proof : The proof is divided into three stages.

The first stage shows that if $P_1, P_2, \ldots, P_n \models Q$, then $\models P_1 \rightarrow (P_2 \rightarrow (\ldots(P_n \rightarrow Q)\ldots))$.
The second stage shows that if $\models \eta$, then $'\eta$, for any formula η (including those looking like what the first stage produced). If $\models \eta$, then η is called a tautology.
The third stage shows that if $'P_1 \rightarrow (P_2 \rightarrow (\ldots(P_n \rightarrow \psi)\ldots))$, then $P_1, P_2, \ldots, P_n \vdash Q$

The first and third stages are the easiest, so let's get them out of the way. We first need to show that if $P_1, P_2, \ldots, P_n \models \psi$, then $\models P_1 \rightarrow (P_2 \rightarrow (\ldots(P_n \rightarrow \psi)\ldots))$. How can we have $\Phi(P_1 \rightarrow (P_2 \rightarrow (\ldots(P_n \rightarrow \psi)\ldots))) = F$ for an interpretation Φ? We would need $\Phi(P_1) = T$ and $\Phi(P_1 \rightarrow (P_1 \rightarrow \ldots(P_1 \rightarrow Q)\ldots)) = F$. We can continue to unravel the formula, and conclude that $\Phi(P_1) = \Phi(P_2) = \ldots = \Phi(P_n) = T$, and $\Phi(Q) = F$. But we know $P_1, P_2, \ldots, P_n \models Q$, so this is impossible, by definition of $\models$. For the third stage, we must take a proof of $'P_1 \rightarrow (P_2 \rightarrow (\ldots(P_n \rightarrow Q)\ldots))$, and from it construct a proof of $P_1, P_2, \ldots P_n \models Q$.

This can be done simply by augmenting the proof of $P_1 \rightarrow (P_2 \rightarrow (\ldots(P_n \rightarrow Q)\ldots))$, with n lines at the beginning introducing $P_1, P_2, \ldots P_n$ as premises, and n lines at the end, each applying $\rightarrow e$ to peel off P_1, then P_1, and so on up to P_1, which leaves ψ as the last line. The result is a proof of $P_1, P_1, \ldots, P_n \models Q$. Technically speaking, both of these stages should be proofs by induction on n, but they are so "iterative" or algorithmic in nature that the induction will only obscure what is going on. This is also true in the second stage, but there we also must do a structural induction on formulas, so we really need to keep things as clear as possible.

The second stage is more complicated than anything we have done so far, though it is not technically difficult. Our goal is to prove that for any formula η, if $\models \eta$, then η. Suppose η has n distinct propositional atoms. (This n is different from the one in the first and third stages.) We are going to have to construct a proof knowing only that each of the 2^n interpretations assign T to η. We do this by constructing 2^n subproofs, one for each interpretation, and then putting them all together. An interpretation Φ assigns T or F to each of the atoms, which we'll call $p_1, p_2, \ldots, p_n$. (You can think of it as a line in a truth table for η.) Relative to Φ, we define $\hat{p}_i$ to be the formula p_i if $\Phi(p_i) = T$; otherwise $\hat{p}_i$ is the formula $\sim p_i$. Hence, for all p_i, $\Phi(\hat{p}_i) = T$

Illustrative Examples

1. **When is a collection of propositional connectives said to be adequate system of connectives? Examine whether the set (,) is an adequate system of connectives**

Solution Let S be a set of connectives. If the set of connectives S is sufficient to express every possible propositional formula, we call it an adequate set of connectives or functionally complete set of connectives.

Let us consider that a propositional formula contains $\vee$ and $\rightarrow$ only. We assume that the propositional function or statement function assumes the value F when the propositional variables takes truth value T. In this case the value of the function cannot be generated since only the connectives $\vee$ and $\rightarrow$ are involved in it. We know that this statement formula assumes true values only when the propositions in it take true values as both $p \vee q$ and $p \rightarrow q$ takes true value when both p and q takes true value T.

Hence, $(\vee, \rightarrow)$ is not an adequate set of connectives

2. **Prove that the set $(\sim, \wedge, \vee)$ is an adequate set of connectives**

Solution We can write $p \rightarrow q \equiv \sim p \vee q$ and

$$
\begin{aligned}
p \leftrightarrow q &\equiv (p \rightarrow q) \wedge (q \rightarrow p) \\
&\equiv (\sim p \vee q) \wedge (\sim q \vee p) \quad [p \rightarrow q \equiv \sim p \vee q] \\
&\equiv ((\sim p \vee q) \wedge \sim q) \vee ((\sim p \vee q) \wedge p) \text{ [Distributive law]} \\
&\equiv (\sim p \wedge \sim q) \vee (q \wedge \sim q) \vee (\sim p \wedge p) \vee (q \wedge p) \text{ [Distributive law]} \\
&\equiv (\sim p \wedge \sim q) \vee F \vee F \vee (q \wedge p) \text{ [Complement law]} \\
&\equiv (\sim p \wedge \sim q) \vee (q \wedge p) \text{ [Identity law]} \\
&\equiv (p \vee q) \vee (\sim p \wedge \sim q) \text{ [Commutative law]}
\end{aligned}
$$

We find that the logical connectives $\rightarrow$ and $\leftrightarrow$ can be written in terms of the connectives in the set $(\sim, \wedge, \vee)$. Hence, the set $(\sim, \wedge, \vee)$ is an adequate set of connectives.

3. **Examine whether the following set of connectives are adequate set of connectives (i) $(\sim, \wedge)$ (ii) $(\sim, \vee)$ (iii) $(\wedge, \vee)$, (iv) $(\sim, \rightarrow)$**

Solution (i). We know that the set $(\sim, \wedge, \vee)$ is an adequate set of connectives.

By De Morgan's law we have $\sim(p \vee q) \equiv \sim p \wedge \sim q$

Also, $(p \vee q) \equiv \sim((\sim p) \wedge (\sim q))$

Hence, the connective $\vee$ can be defined in terms of the connectives $\sim$ and $\wedge$.

Hence, the set $(\sim, \wedge)$ is an adequate set of connectives.

(ii) We know that the set $(\sim, \wedge, \vee)$ is an adequate set of connectives.
By De Morgan's law we have $\sim(p \wedge q) \equiv \sim p \vee \sim q$
Also, $(p \wedge q) \equiv \sim(\sim p) \vee (\sim q))$
Hence, the connective $\wedge$ can be defined in terms of the connectives $\sim$ and $\vee$.
Hence, the set $(\sim, \vee)$ is an adequate set of connectives.

(iii) The set $(\wedge, \vee)$ is not adequate set of connectives since $\sim$ can not be defined by $\wedge$ and $\vee$.

(iv) We know that the set $(\sim, \wedge, \vee)$ is an adequate set of connectives.
We can write $p \vee q \equiv \sim p \to q$ and $p \wedge q \equiv \sim(p \to \sim q)$
Thus, the connective $\wedge$ and $\vee$ can be defined in terms of the connective $\to$.
Hence, the set $(\sim, \to)$ is an adequate set of connectives.

4 **Disjunctive normal form (DNF) of the formula $p \leftrightarrow q$ is**
(i) $(p \wedge q) \vee (p \wedge \sim q)$ **(ii)** $(p \wedge \sim q) \vee (p \wedge q)$ **(iii)** $(\sim p \wedge q) \vee (p \wedge q)$
(iv) $(p \wedge q) \vee (\sim p \wedge \sim q)$ [CU, 2021]

Solution $p \leftrightarrow q \equiv (p \wedge q) \vee (\sim p \wedge \sim q)$ [Refer to table 2.7]
The option number (iv) is the correct option.

5 **Find the disjunctive normal form (DNF) of the following statements :**
(i) $\sim(\sim(p \leftrightarrow q) \wedge r)$ **(ii)** $p \vee (\sim p \to (q \vee (q \to \sim r)))$
(iii) $p \wedge \sim(q \wedge r) \vee (p \to q)$ **(iv)** $p \to (p \to q) \wedge (\sim(\sim q \vee \sim p))$
(v) $p \wedge (p \to q)$ **(vi)** $(p \wedge \sim(q \vee r)) \vee ((p \wedge q) \vee \sim r) \wedge p$

Solution (i) $\sim(\sim(p \leftrightarrow q) \wedge r) \equiv \sim[\sim\{(p \wedge q) \vee (\sim p \wedge \sim q)\} \wedge r]$ [Refer to table 2.7]
$\equiv \sim[\{\sim(p \wedge q) \wedge \sim(\sim p \wedge \sim q)\} \wedge r]$
$\equiv \sim[\{(\sim p \vee \sim q) \wedge (p \vee q)\} \wedge r]$ [De Morgan's law]
$\equiv \sim[\{(\sim p \wedge p) \vee (\sim p \wedge q) \vee (\sim q \wedge p) \vee (\sim q \wedge q)\} \wedge r]$, by extended distributive law
$\equiv \sim[\{(\sim p \wedge q) \vee (\sim q \vee p)\} \wedge r]$, by identity and complement law
$\equiv \sim[\{(\sim p \vee \sim q) \wedge (\sim p \vee p) \wedge (q \wedge \sim q) \wedge (q \vee p)\} \wedge r]$, by extended distributive law
$\equiv \sim[\{(\sim p \vee \sim q) \wedge (q \vee p)\} \wedge r]$, by identity and complement law
$\equiv \sim(p \vee q) \vee \sim(\sim p \vee \sim q) \vee \sim r$, by commutative and De Morgan's law
$\equiv (\sim p \wedge \sim q) \vee (p \wedge q) \vee \sim r$, by De Morgan's law
This is the required DNF

(ii) $p \vee (\sim p \to (q \vee (q \to \sim r))) \equiv p \vee (\sim p \to (q \vee (\sim q \vee \sim r)))$ [as $p \to q \equiv \sim p \vee q$]
$\equiv p \vee (p \vee (q \vee (\sim q \vee \sim r)))$ [as $p \to q \equiv \sim p \vee q$]
$\equiv p \vee p \vee q \vee \sim q \vee \sim r$
$\equiv p \vee q \vee \sim q \vee \sim r$, by idempotent law
This is the required DNF

(iii) $p \wedge \sim(q \wedge r) \vee (p \to q) \equiv p \wedge \sim(q \wedge r) \vee (\sim p \vee q)$ [as $p \to q \equiv \sim p \vee q$]
$\equiv (p \wedge (\sim q \vee \sim r)) \vee (\sim p \vee q)$, by De Morgan's law
$\equiv ((p \wedge \sim q) \vee (p \wedge \sim r)) \vee (\sim p \vee q)$, by distributive law
$\equiv (p \wedge \sim q) \vee (p \wedge \sim r) \vee (\sim p \vee q)$,
This is the required DNF

(iv) $p \to (p \to q) \wedge (\sim(\sim q \vee \sim p)) \equiv p \to ((p \to q) \wedge \sim(\sim q \vee \sim p))$
$\equiv \sim p \vee ((p \to q) \wedge (\sim(\sim q \vee \sim p))$ [as $p \to q \equiv \sim p \vee q$]
$\equiv \sim p \vee ((p \to q) \wedge (\sim \sim q \wedge \sim\sim p))$, by De Morgan's law
$\equiv \sim p \vee ((p \to q) \wedge (q \wedge p))$
$\equiv \sim p \vee ((\sim p \vee q) \wedge (q \wedge p))$ [$p \to q \equiv \sim p \vee q$]
$\equiv \sim p \vee ((\sim p \wedge (q \wedge p)) \vee ((q \wedge (q \wedge p))$, by distributive law
$\equiv \sim p \vee ((\sim p \wedge (p \wedge q) \vee (q \wedge q \wedge p))$, By associative and commutative
$\equiv \sim p \vee ((\sim p \wedge p) \wedge q) \vee (q \wedge p))$, by associative and idempotent law
$\equiv \sim p \vee (F \wedge q) \vee (q \wedge p))$, by complement law
$\equiv \sim p \vee \{F \vee (q \wedge p)\}$, by dominant law
$\equiv \sim p \vee (q \wedge p)$, by identity law
This is the required DNF

(v) $p \wedge (p \to q) \equiv p \wedge (\sim p \vee q)$ [as $p \to q \equiv \sim p \vee q$]
$\equiv (p \wedge \sim p) \vee (p \wedge q)$, by distributive law
This is the required DNF

(vi) $(p \wedge \sim(q \vee r)) \vee (((p \wedge q) \vee \sim r) \wedge p$
$\equiv (p \wedge \sim(\sim q \wedge \sim r)) \vee ((p \wedge q) \wedge p) \vee (\sim r \wedge p))$, by De Morgan's and distribu
$\equiv (p \wedge q \wedge \sim r) \vee (p \wedge q) \vee (p \wedge \sim r)$,
by associative, commutative and idempote
This is the required DNF

6. **Find the conjunctive normal form (CNF) of the following statem**

(i) $\sim(p \vee q) \leftrightarrow (p \wedge q)$ **(ii)** $p \wedge (p \to q)$

(iii) $(q \vee (p \wedge r)) \wedge \sim((p \vee r) \wedge q)$ **(iv)** $(p \wedge \sim(q \wedge r)) \vee (p \to q)$

Solution (i) $\sim(p \vee q) \leftrightarrow (p \wedge q) \equiv (\sim(p \vee q) \to (p \wedge q)) \wedge ((p \wedge q) \to \sim(p \vee$
[as $p \leftrightarrow q \equiv \sim((p \to q) \wedge (q \to$
$\equiv ((\sim(p \vee q) \vee (p \wedge q)) \wedge (\sim(p \wedge q) \vee \sim(p \vee q))$, [as $p \to q \equiv \sim p \vee q$]
$\equiv ((p \vee q) \vee (p \wedge q)) \wedge \sim((p \wedge q) \wedge (p \vee q))$, by De Morgan's law
$\equiv (((p \vee q) \vee p) \wedge ((p \vee q) \vee q))) \wedge \sim((p \wedge (q \wedge (p \vee q)))$,
by distributive and associa
$\equiv ((q \vee p) \vee p) \wedge (p \vee (q \vee q)) \wedge \sim(p \wedge ((q \wedge p) \vee (q \wedge q)))$,
by distributive, associative and commut
$\equiv ((q \vee (p \vee p)) \wedge (p \vee q) \wedge \sim(p \wedge ((q \wedge p) \vee q)$,
by associative and idempo

$\equiv (q \vee p) \wedge (p \vee q) \wedge \sim(p \wedge (q$
$\equiv (p \vee q) \wedge (p \vee q) \wedge \sim(p \wedge (p$
$\equiv (p \vee q) \wedge \sim((p \wedge p) \wedge q) \vee (p$
$\equiv (p \vee q) \wedge \sim((p \wedge q) \vee (p \wedge q$
$\equiv (p \vee q) \wedge \sim(p \wedge q)$, by idem
$\equiv (p \vee q) \wedge (\sim p \vee \sim q)$, by De
This is the required CNF

(ii) $p \wedge (p \to q) \equiv p \wedge (\sim p \vee q)$ [as
This is the required CNF

(iii) $(q \vee (p \wedge r)) \wedge \sim((p \vee r) \wedge q) \equiv$
$\equiv (q \vee (p \wedge r)) \wedge ((\sim p \wedge \sim r)$
$\equiv ((q \vee p) \wedge (q \vee r)) \wedge ((\sim p$
$\equiv (q \vee p) \wedge (q \vee r) \wedge (\sim p \vee$
This is the required CNF

(iv) $(p \wedge \sim(q \wedge r)) \vee (p \to q) \equiv (p \wedge$
$\equiv (p \wedge (\sim q \vee \sim r)) \vee (\sim p \vee q)$
$\equiv ((p \wedge \sim q) \vee (p \wedge \sim r)) \vee (\sim$
$\equiv (p \vee p) \wedge (p \vee \sim r) \wedge (\sim q \vee$
$\equiv (p \vee p) \wedge (p \vee \sim r)) \wedge (\sim q$
$\equiv (p \vee p) \wedge (p \vee \sim r)) \wedge (\sim q$
$\equiv p \wedge (p \vee \sim r) \wedge (\sim q \vee p)$ by
$\equiv p \wedge (p \vee \sim r) \wedge (p \vee \sim q)$, b
This is the required CNF

7. **Without using truth tab**
(PDNF) of the following
(i) $p \vee \sim q$
(iii) $(p \wedge q) \vee (\sim p \wedge q) \vee$

Solution (i) $p \vee \sim q \equiv p \wedge (q \vee$
$\equiv (p \wedge q) \vee (p \wedge \sim q) \vee (\sim q \wedge$
$\equiv (p \wedge q) \vee (p \wedge \sim q) \vee (\sim q \wedge$
This is the required PDNF

(ii) $(\sim p \to q) \wedge (q \leftrightarrow p) \equiv (p \vee q)$
[as
$\equiv (p \vee q) \wedge ((\sim q \vee p) \wedge (\sim p$

$\equiv (q \vee p) \wedge (p \vee q) \wedge \sim (p \wedge (q \wedge p)) \vee (p \wedge q)$,
by distributive and idempotent laws
$\equiv (p \vee q) \wedge (p \vee q) \wedge \sim (p \wedge (p \wedge q)) \vee (p \wedge q)$, by commutative law
$\equiv (p \vee q) \wedge \sim ((p \wedge p) \wedge q) \vee (p \wedge q)$, by idempotent and associative laws
$\equiv (p \vee q) \wedge \sim ((p \wedge q) \vee (p \wedge q))$, by idempotent law
$\equiv (p \vee q) \wedge \sim (p \wedge q)$, by idempotent law
$\equiv (p \vee q) \wedge (\sim p \vee \sim q)$, by De Morgan's law
This is the required CNF

(ii) $p \wedge (p \to q) \equiv p \wedge (\sim p \vee q)$ [as $p \to q \equiv \sim p \vee q$]
This is the required CNF

(iii) $(q \vee (p \wedge r)) \wedge \sim ((p \vee r) \wedge q) \equiv (q \vee (p \wedge r)) \wedge (\sim (p \vee r) \vee \sim q)$, by De Morgan's law
$\equiv (q \vee (p \wedge r)) \wedge ((\sim p \wedge \sim r) \vee \sim q))$, by De Morgan's law
$\equiv ((q \vee p) \wedge (q \vee r)) \wedge ((\sim p \vee \sim q) \wedge (\sim r \vee \sim q))$, by distributive law
$\equiv (q \vee p) \wedge (q \vee r) \wedge (\sim p \vee \sim q) \wedge (\sim r \vee \sim q)$,
This is the required CNF

(iv) $(p \wedge \sim (q \wedge r)) \vee (p \to q) \equiv (p \wedge \sim (q \wedge r)) \vee (\sim p \vee q)$ [as $p \to q \equiv \sim p \vee q$]
$\equiv (p \wedge (\sim q \vee \sim r)) \vee (\sim p \vee q)$, by De Morgan's law
$\equiv ((p \wedge \sim q) \vee (p \wedge \sim r)) \vee (\sim p \vee q)$, by distributive law
$\equiv (p \vee p) \wedge (p \vee \sim r) \wedge (\sim q \vee p) \wedge (\sim q \vee \sim r) \vee (\sim p \vee q)$,
by extended distributive law
$\equiv (p \vee p) \wedge (p \vee \sim r)) \wedge (\sim q \vee p) \wedge (\sim p \vee q \vee \sim q \vee \sim r)$
by associative and commutative law
$\equiv (p \vee p) \wedge (p \vee \sim r)) \wedge (\sim q \vee p) \wedge (\sim p \vee \mathrm{T} \vee \sim r)$, by complement law
$\equiv p \wedge (p \vee \sim r) \wedge (\sim q \vee p)$ by idempotent, identity and dominat laws
$\equiv p \wedge (p \vee \sim r) \wedge (p \vee \sim q)$, by commutative law
This is the required CNF

7. Without using truth tables, find the principal disjunctive normal form (PDNF) of the following statements :

(i) $p \vee \sim q$ **(ii)** $(\sim p \to \sim q) \wedge (q \leftrightarrow p)$

(iii) $(p \wedge q) \vee (\sim p \wedge q) \vee (q \wedge r)$ **(iv)** $(q \vee (p \wedge r) \wedge \sim ((p \vee r) \wedge q)$

Solution (i) $p \vee \sim q \equiv p \wedge (q \vee \sim q) \vee \sim q \wedge (p \vee \sim p)$, by complement law
$\equiv (p \wedge q) \vee (p \wedge \sim q) \vee (\sim q \wedge p) \vee (\sim q \wedge \sim p)$, by distributive law
$\equiv (p \wedge q) \vee (p \wedge \sim q) \vee (\sim q \wedge \sim p)$, by commutative and idempotent laws
This is the required PDNF

(ii) $(\sim p \to q) \wedge (q \leftrightarrow p) \equiv (p \vee q) \wedge ((q \to p) \wedge (p \to q))$,
[as $p \to q \equiv \sim p \vee q$ and $p \leftrightarrow q \equiv ((p \to q) \wedge (q \to p))$]
$\equiv (p \vee q) \wedge ((\sim q \vee p) \wedge (\sim p \vee q))$, [as $p \to q \equiv \sim p \vee q$]

$\equiv (p \vee q) \wedge ((\sim q \vee \sim p) \wedge (\sim q \vee q)) \wedge ((p \wedge \sim p) \vee (p \wedge q))$, by extended distri

$\equiv (p \vee q) \wedge ((\sim q \vee \sim p) \wedge T) \wedge (F \vee (p \wedge q))$,

by complement laws, T is tautology and F is Contra

$\equiv (p \vee q) \wedge ((p \wedge q) \vee (\sim q \vee \sim p))$, by identity and commutative laws

$\equiv (p \vee q) \wedge ((p \wedge q) \vee \sim (p \vee q))$, by commutative and De Morgan's

$\equiv ((p \vee q) \wedge ((p \vee q)) \vee ((p \vee q) \wedge \sim (p \vee q))$, by distributive law

$\equiv ((p \vee q) \wedge ((p \wedge q)) \vee F$, by complement law

$\equiv (p \wedge (p \wedge q)) \vee ((q \wedge (p \wedge q))$, by distributive and identity laws

$\equiv ((p \wedge p) \wedge q)) \vee ((q \wedge (q \wedge p))$, by associative and commutative la

$\equiv (p \wedge q) \vee ((q \wedge q) \wedge p))$, by idempotent and associative law

$\equiv (p \wedge q) \vee (q \wedge p))$, by idempotent law

$\equiv (p \wedge q) \vee (p \wedge q))$, by commutative law

$\equiv (p \wedge q)$, by idempotent law

This is the required PDNF

(iii) $(p \wedge q) \vee (\sim p \wedge q) \vee (q \wedge r)$

$\equiv ((p \wedge q) \wedge (r \vee \sim r)) \vee ((\sim p \wedge q) \wedge (r \vee \sim r)) \vee ((q \wedge r) \wedge (p \vee \sim p))$

by compleme

This is the required DNF but not PDNF

$\equiv ((\sim p \wedge q \wedge r) \vee (p \wedge q \wedge \sim r)) \vee ((\sim p \wedge q \wedge r) \vee ((\sim p \wedge q \wedge \sim r))$

$\vee ((q \wedge r \wedge p) \vee (q \wedge r \wedge \sim p))$, by distribu

$\equiv (p \wedge q \wedge r) \vee (p \wedge q \wedge \sim r) \vee (\sim p \wedge q \wedge r) \vee ((\sim p \wedge q \wedge \sim r)$,

[Deleting repetitions of identical m

This is the required PDNF

(iv) $(q \vee (p \wedge r) \wedge \sim((p \vee r) \wedge q) \equiv (q \vee (p \wedge r)) \wedge (\sim (p \vee r) \vee \sim q)$, by De Morg

$\equiv (q \vee (p \wedge r)) \wedge ((\sim p \wedge \sim r) \vee \sim q)$, by De Morgan's law

$\equiv (q \wedge \sim p \wedge \sim r) \vee (q \wedge \sim q) \vee (p \wedge r \wedge \sim p \wedge \sim r) \vee (p \wedge r \wedge \sim q)$,

by extended distribu

$\equiv (\sim p \wedge q \wedge \sim r) \vee F \vee F \vee (p \wedge \sim q \wedge r)$, by complement and commu

$\equiv (\sim p \wedge q \wedge \sim r) \vee (p \wedge \sim q \wedge r)$, by identity laws

This is the required PDNF

8. **Without using truth tables, find the principal conjunctive norma (PCNF) of the following statements:**

(i) $p \leftrightarrow q$ **(ii)** $(p \wedge q) \vee (\sim p \wedge q \wedge r)$

(iii) $(p \vee q) \wedge (r \vee \sim p) \wedge (q \vee \sim r)$ **(iv)** $(p \rightarrow (q \wedge r)) \wedge (\sim p \rightarrow (\sim q \wedge$

Solution **(i)** $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$ [Refer to table 2.7]

$\equiv (p \wedge q) \vee (\sim p \wedge \sim q)$ [Refer to table 2.7]

This is in PDNF

Again, PDNF of $p \leftrightarrow q \equiv (\sim p \wedge q) \vee$

So, $(p \leftrightarrow q) \equiv \sim\sim (p \leftrightarrow q)$

$\equiv \sim (\sim p \wedge q) \wedge \sim (p \wedge \sim$

$\equiv (p \vee \sim q) \wedge (\sim p \vee q)$,

Which is same as $p \leftrightarrow q \equiv (p \rightarrow q)$

(ii) $(p \wedge q) \vee (\sim p \wedge q \wedge r) \equiv ((p \wedge q)$

$\equiv (p \vee \sim p) \wedge (q \vee \sim p) \wedge (p \vee q) \wedge$

$\equiv T \wedge (\sim p \vee q) \wedge (p \vee q) \wedge q \wedge ($

$\equiv ((\sim p \vee q) \vee (r \wedge \sim r)) \wedge ((p \vee q)$

$\wedge (p \vee r$

$\equiv (\sim p \vee q \vee r) \wedge (\sim p \vee q \vee \sim r) \wedge$

$(q \vee p) \wedge (q \vee \sim p) \wedge (p$

$\equiv (\sim p \vee q \vee r) \wedge (\sim p \vee q \vee \sim r) \wedge$

$(p \vee \sim$

$\equiv (\sim p \vee q \vee r) \wedge (\sim p \vee q \vee \sim r) \wedge$

This is the required PCNF

(iii) $(p \vee q) \wedge (r \vee \sim p) \wedge (q \vee \sim r)$

$\equiv ((p \vee q) \vee (r \wedge \sim r)) \wedge ((\sim p$

$\equiv (p \vee q \vee r) \wedge (p \vee q \vee \sim r)$

$\wedge (p \vee q \vee \sim r) \wedge (\sim$

$\equiv (p \vee q \vee r) \wedge (p \vee q \vee \sim r)$

This is the required PCNF

(iv) $(p \rightarrow (q \wedge r)) \wedge (\sim p \rightarrow (\sim q \wedge \sim$

$\equiv (\sim p \vee (q \wedge r)) \wedge (p \vee (\sim q$

$\equiv ((\sim p \vee q) \wedge (\sim p \vee r)) \wedge ((\sim$

$\equiv ((\sim p \vee q) \vee (r \wedge \sim r)) \wedge ((\sim$

$\equiv (\sim p \vee q \vee r) \wedge (\sim p \vee q \vee \sim$

$\wedge (p \vee \sim q \vee \sim r)$

$\equiv (p \vee \sim q \vee r) \wedge (p \vee \sim q \vee$

This is the required PCNF

Again, PDNF of $p \leftrightarrow q \equiv (\sim p \wedge q) \vee (p \wedge \sim q)$, [remaining minterms]

So, $(p \leftrightarrow q) \equiv \sim\sim (p \leftrightarrow q)$ ... (1)

$\equiv \sim (\sim p \wedge q) \wedge \sim (p \wedge \sim q)$, by De Morgan's law

$\equiv (p \vee \sim q) \wedge (\sim p \vee q)$, by De Morgan's law

Which is same as $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$

(ii) $(p \wedge q) \vee (\sim p \wedge q \wedge r) \equiv ((p \wedge q) \vee \sim p) \wedge ((p \wedge q) \vee q) \wedge ((p \wedge q) \vee r)$,

by distributive law

$\equiv (p \vee \sim p) \wedge (q \vee \sim p) \wedge (p \vee q) \wedge (q \vee q) \wedge (p \vee r) \wedge (q \vee r)$, by distributive law

$\equiv T \wedge (\sim p \vee q) \wedge (p \vee q) \wedge q \wedge (p \vee r) \wedge (q \vee r)$,

by identity, commutative and idempotent laws

$\equiv ((\sim p \vee q) \vee (r \wedge \sim r)) \wedge ((p \vee q) \vee (r \wedge \sim r)) \wedge q \vee (p \wedge \sim p)$

$\wedge (p \vee r) \vee (q \wedge \sim q) \wedge (q \vee r) \vee (p \wedge \sim p)$, by identity law

$\equiv (\sim p \vee q \vee r) \wedge (\sim p \vee q \vee \sim r) \wedge (p \vee q \vee r) \wedge (p \vee q \vee \sim r) \wedge$

$(q \vee p) \wedge (q \vee \sim p) \wedge (p \vee r \vee q) \wedge (p \vee r \vee \sim q) \wedge (q \vee r \vee p) \wedge (q \vee r \vee \sim p)$,

by distributive law

$\equiv (\sim p \vee q \vee r) \wedge (\sim p \vee q \vee \sim r) \wedge (p \vee q \vee r) \wedge (p \vee q \vee \sim r) \wedge$

$(p \vee \sim q \vee r) \wedge ((q \vee p) \vee (r \wedge \sim r)) \wedge ((q \vee \sim p) \vee (r \wedge \sim r))$

[Omitting repetitions]

$\equiv (\sim p \vee q \vee r) \wedge (\sim p \vee q \vee \sim r) \wedge (p \vee q \vee r) \wedge (p \vee q \vee \sim r) \wedge (p \vee \sim q \vee r)$

[Omitting repetitions]

This is the required PCNF

(iii) $(p \vee q) \wedge (r \vee \sim p) \wedge (q \vee \sim r)$

$\equiv ((p \vee q) \vee (r \wedge \sim r)) \wedge ((\sim p \vee r) \vee (q \wedge \sim q)) \wedge ((q \vee \sim r) \vee (p \wedge \sim p))$

$\equiv (p \vee q \vee r) \wedge (p \vee q \vee \sim r) \wedge (\sim p \vee q \vee r) \wedge (\sim p \vee \sim q \vee r)$

$\wedge (p \vee q \vee \sim r) \wedge (\sim p \vee q \vee \sim r)$, by distributive and commutative law

$\equiv (p \vee q \vee r) \wedge (p \vee q \vee \sim r) \wedge (\sim p \vee q \vee r) \wedge (\sim p \vee \sim q \vee r) \wedge (\sim p \vee q \vee \sim r)$,

by idempotent law

This is the required PCNF

(iv) $(p \rightarrow (q \wedge r)) \wedge (\sim p \rightarrow (\sim q \wedge \sim r))$

$\equiv (\sim p \vee (q \wedge r)) \wedge (p \vee (\sim q \wedge \sim r))$ [as $p \rightarrow q \equiv \sim p \vee q$]

$\equiv ((\sim p \vee q) \wedge (\sim p \vee r)) \wedge ((p \vee \sim q) \wedge (p \vee \sim r))$, by distributive law

$\equiv ((\sim p \vee q) \vee (r \wedge \sim r)) \wedge ((\sim p \vee r) \vee (q \wedge \sim q)) \wedge ((p \vee \sim q) \vee (r \wedge \sim r))$

$\wedge ((p \vee \sim r) \vee (q \wedge \sim q))$,

$\equiv (\sim p \vee q \vee r) \wedge (\sim p \vee q \vee \sim r) \wedge (\sim p \vee r \vee q) \wedge (\sim p \vee r \vee \sim q) \wedge (p \vee \sim q \vee r)$

$\wedge (p \vee \sim q \vee \sim r) \wedge (p \vee \sim r \vee q) \wedge (p \vee \sim r \vee \sim q)$, by distributive law

$\equiv (p \vee \sim q \vee r) \wedge (p \vee \sim q \vee \sim r) \wedge (\sim p \vee q \vee \sim r) \wedge (\sim p \vee q \vee r) \wedge (\sim p \vee \sim q \vee r)$

$\wedge (\sim p \vee q \vee \sim r)$ [Omitting repetitions]

This is the required PCNF

9. **What do you mean by principal disjunctive normal form (P... Obtain the principal disjunctive normal form of the following**
(i) $p \to q$ **(ii)** $q \vee (p \vee \sim q)$

Solution A compound proposition consisting of disjunctions of minterm... variables only and equivalent to a given formula is known as its principal dis... normal form (PDNF). PDNF is also called 'sum of product canonical form'.

(i) The truth table corresponding to $p \to q$ is

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

In the last column of the above truth table, "T" corresponds to the first ... and fourth row. The conjunctions of literals (minterms) corresponding to ... rows are $(p \wedge q)$, $(\sim p \wedge q)$ and $(\sim p \wedge \sim q)$.
Now the disjunction of the three minterms is $(p \wedge q) \vee (\sim p \wedge q) \vee (\sim p \wedge \sim q)$ ... is the required PDNF

(ii) The truth table corresponding to $q \vee (p \vee \sim q)$ is

p	q	$\sim q$	$p \vee \sim q$	$q \vee (p \vee \sim q)$
T	T	F	T	T
T	F	T	T	T
F	T	F	F	T
F	F	T	T	T

In the last column of the above truth table, "T" corresponds to all the ... The conjunctions of literals (minterms) corresponding to the rows are ... $(p \wedge \sim q)$, $(\sim p \wedge q)$ and $(\sim p \wedge \sim q)$
Now the disjunction of the three minterms is
$(p \wedge q) \vee (p \wedge \sim q) \vee (\sim p \wedge q) \vee (\sim p \wedge \sim q)$
which is the required PDNF

10. **Using truth table, find the principal disjunctive normal form (P... and principal conjunctive normal form (PCNF) of the follo... statements :**
(i) $(\sim p \wedge \sim q) \to (p \leftrightarrow \sim q)$ **(ii)** $p \vee (\sim p \to (q \vee (\sim q \to r)))$
(iii) $(p \to (q \wedge r)) \wedge (\sim p \to (\sim q \to \sim r))$ **(iv)** $p \wedge ((q \leftrightarrow r) \vee (r \leftrightarrow p))$

Solution **(i)** The truth table corresponding to $(\sim p \wedge \sim q) \to (p \leftrightarrow \sim q)$ is

p	q	$\sim p$	$\sim q$	$\sim p \wedge \sim q$	$p \leftrightarrow \sim q$	$(\sim p \wedge \sim q) \to (p \leftrightarrow \sim q)$
T	T	F	F	F	F	T
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	T	F	F

In the last column of the above tr... and third rows. The conjunctions ... second and third rows are $(p \wedge q)$, $(p$ ... three minterms is
$(p \wedge q) \vee (p \wedge \sim q) \vee (\sim p \wedge q)$
which is the required PDNF.
Again, in the last column of the ... row. We take p and q correspondi... corresponding to "F" in the last c... and q i.e., $p \vee q$.
This is the only maxterm whic...

(ii) The truth table corresponding...

p	q	r	$\sim p$	$\sim q$	$\sim q \to r$	$q \vee (\sim q$
T	T	T	F	F	T	T
T	T	F	F	F	T	T
T	F	T	F	T	T	T
T	F	F	F	T	F	F
F	T	T	T	F	T	T
F	T	F	T	F	T	T
F	F	T	T	T	T	T
F	F	F	T	T	F	F

"T" corresponds to all the ro... above truth table. The conju... these rows are $(p \wedge q \wedge r)$, $(p$ ... $(\sim p \wedge q \wedge \sim r)$ and $(\sim p \wedge \sim q \wedge$ ... $(p \wedge q \wedge r) \vee (p \wedge q \wedge \sim r) \vee (p$

which is the required PDNF.
Again, in the last column of ... row. We take p, q and r corre... generate the disjunction of ... This is the only maxterm wh...

(iii) The truth table correspondi...

p	q	r	$\sim p$	$\sim q$	$\sim r$	$q \wedge r$
T	T	T	F	F	F	T
T	T	F	F	F	T	F
T	F	T	F	T	F	F
T	F	F	F	T	T	F
F	T	T	T	F	F	T
F	T	F	T	F	T	F
F	F	T	T	T	F	F
F	F	F	T	T	T	F

In the last column of the above truth table, "T" corresponds to the first, second and third rows. The conjunctions of literals (minterms) corresponding to first, second and third rows are $(p \wedge q)$, $(p \wedge \sim q)$ and $(\sim p \wedge q)$. Now the disjunction of the three minterms is

$$(p \wedge q) \vee (p \wedge \sim q) \vee (\sim p \wedge q)$$

which is the required PDNF.

Again, in the last column of the above truth table, "F" corresponds to the last row. We take p and q corresponding to "F" and "F" respectively of the last row corresponding to "F" in the last column. Then we generate the disjunction of p and q i.e., $p \vee q$.

This is the only maxterm which is the required PCNF.

(ii) The truth table corresponding to $p \vee (\sim p \rightarrow (q \vee (\sim q \rightarrow r))$ is

p	q	r	$\sim p$	$\sim q$	$\sim q \rightarrow r$	$q \vee (\sim q \rightarrow r)$	$\sim p \rightarrow (q \vee (\sim q \rightarrow r))$	$p \vee (\sim p \rightarrow (q \vee (\sim q \rightarrow r))$
T	T	T	F	F	T	T	T	T
T	T	F	F	F	T	T	T	T
T	F	T	F	T	T	T	T	T
T	F	F	F	T	F	F	T	T
F	T	T	T	F	T	T	T	T
F	T	F	T	F	T	T	T	T
F	F	T	T	T	T	T	T	T
F	F	F	T	T	F	F	F	F

"T" corresponds to all the rows of the last column except the last one of the above truth table. The conjunctions of literals (minterms) corresponding to these rows are $(p \wedge q \wedge r)$, $(p \wedge q \wedge \sim r)$, $(p \wedge \sim q \wedge r)$, $(p \wedge \sim q \wedge \sim r)$, $(\sim p \wedge q \wedge r)$, $(\sim p \wedge q \wedge \sim r)$ and $(\sim p \wedge \sim q \wedge r)$. Now the disjunction of these minterms is

$$(p \wedge q \wedge r) \vee (p \wedge q \wedge \sim r) \vee (p \wedge \sim q \wedge r) \vee (p \wedge \sim q \wedge \sim r) \vee (\sim p \wedge q \wedge r) \vee (\sim p \wedge q \wedge \sim r) \vee (\sim p \wedge \sim q \wedge r).$$

which is the required PDNF.

Again, in the last column of the above truth table, "F" corresponds to the last row. We take p, q and r corresponding to all three "F" of the last row. Then we generate the disjunction of p, q and r i.e., $(p \vee q \vee r)$.

This is the only maxterm which is the required PCNF.

(iii) The truth table corresponding to $(p \rightarrow (q \wedge r)) \wedge (\sim p \rightarrow (\sim q \rightarrow \sim r))$ is

p	q	r	$\sim p$	$\sim q$	$\sim r$	$q \wedge r \equiv a$	$p \rightarrow a \equiv b$	$\sim q \wedge \sim r \equiv c$	$\sim p \rightarrow c \equiv d$	$b \wedge d$
T	T	T	F	F	F	T	T	F	T	T
T	T	F	F	F	T	F	F	F	T	F
T	F	T	F	T	F	F	F	F	T	F
T	F	F	F	T	T	F	F	F	T	F
F	T	T	T	F	F	T	T	F	F	F
F	T	F	T	F	T	F	T	F	F	F
F	F	T	T	T	F	F	T	T	T	T
F	F	F	T	T	T	F	T			

"T" corresponds to the first and last rows of the last column of the ... table. The conjunctions of literals (minterms) corresponding to the ... are $(p \wedge q \wedge r) \vee (\sim p \wedge \sim q \wedge \sim r)$

$(p \wedge q \wedge r) \vee (\sim p \wedge \sim q \wedge \sim r)$

which is the required PDNF

Again, in the last column of the above truth table, "F" corresponds ... rows except the first and last rows. We take p, q and r corresponding ... "F". The disjunctions of literals (maxterms) corresponding to these ... $(\sim p \vee \sim q \vee r)$, $(\sim p \vee q \vee \sim r)$, $(\sim p \vee q \vee r)$, $(p \vee \sim q \vee \sim r)$, $(p \vee \sim q \vee r)$ and $(p \vee$ ...

Now the conjunction of these maxterms is

$(\sim p \vee \sim q \vee r) \wedge (\sim p \vee q \vee \sim r) \wedge (\sim p \vee q \vee r) \wedge (p \vee \sim q \vee \sim r) \wedge (p \vee \sim q \vee r) \wedge (p \vee$ ...

which is the required PCNF

(iv) The truth table corresponding to $p \wedge ((q \leftrightarrow r) \vee (r \leftrightarrow p))$ is

p	q	r	$q \leftrightarrow r$	$r \leftrightarrow p$	$(q \leftrightarrow r) \vee (r \leftrightarrow p)$	$p \wedge ((q \leftrightarrow r) \vee (r \leftrightarrow$ ...
T	T	T	T	T	T	T
T	T	F	F	F	F	F
T	F	T	F	T	T	T
T	F	F	T	F	T	T
F	T	T	T	F	T	F
F	T	F	F	T	T	F
F	F	T	F	F	F	F
F	F	F	T	T	T	F

"T" corresponds to the first, third and forth rows of the last column ... above truth table. The conjunctions of literals (minterms) correspond... these rows are $(p \wedge q \wedge r)$, $(p \wedge \sim q \wedge r)$ and $(p \wedge \sim q \wedge \sim r)$. Now the disjun... these minterms is

$(p \wedge q \wedge r) \vee (p \wedge \sim q \wedge r) \vee (p \wedge \sim q \wedge \sim r)$

which is the required PDNF

Again, in the last column of the above truth table, "F" corresponds to ... rows except the first, third and fourth rows. We take p, q and r corresp... to all these "F". The disjunctions of literals (maxterms) corresponding t... rows are $(\sim p \vee \sim q \vee r)$, $(p \vee \sim q \vee \sim r)$, $(p \vee \sim q \vee r)$, $(p \vee q \vee \sim r)$ and $(p \vee q$ ...

Now the conjunction of these maxterms is

$(\sim p \vee \sim q \vee r) \wedge (p \vee \sim q \vee \sim r) \wedge (p \vee \sim q \vee r) \wedge (p \vee q \vee \sim r) \wedge (p \vee q \vee r$ ...

which is the required PCNF

11. **Let B be the set of all positive divisor of 30. Let us define**

$a + b$ = l.c.m of a & b; $a \cdot b$ = g.c.d. of a & b; $a' = \frac{30}{a}$

Examine whether (B, +, ·, ′) is a Boolean algebra or not.

Solution Here, B = {1, 2, 3, 5, 6, 10, 15, 30}

Now, by given definitions

$5 + 6$ = l.c.m. of 5 & 6 = 30 ∈ B

$5 \cdot 6$ = g.c.d. of 5 & 6 = 1 ∈ B

Same results hold for other elements also.

Thus 'B' is closed under '+' and '·'

(a) (i) We have

$x + y$ = l.c.m. of x & y = l.c.m. of y & x =

(ii) Here $x \cdot y$ = g.c.d. of x & y = g.c.d. of y &

So, '+' and '·' are commutative.

(b) (i) We have

$5 + 3 \cdot 6 = 5 + 3 = 15$

$(5 + 3) \cdot (5 + 6) = 15 \cdot 30 = 15$

Thus, $5 + 3 \cdot 6 = (5 + 3) \cdot (5 + 6)$

Similarly result holds for other elem...

i.e. $x + y \cdot z = (x + y) \cdot (x + z)$, $\forall x, y, z \in$ ...

(ii) We have

$3 \cdot (10 + 15) = 3 \cdot 30 = 3$

$(3 \cdot 10) + (3 \cdot 15) = 1 + 3 = 3$

$\therefore 3 \cdot (10 + 15) = (3 \cdot 10) + (3 \cdot 15)$

Similarly result holds for other ele...

i.e. $x \cdot (y + z) = x \cdot y + x \cdot z$, $\forall x, y, z \in$ B

Thus, each of the binary operation...

(c) (i) We have

$5 + 1 = 5$

i.e. $x + 1$ = l.c.m. of x & $1 = x$, $\forall x \in$ ...

So, 1 is the identity element und...

(ii) Now, $x \cdot 30$ = g.c.d. of x & $30 = x$, ...

So, 30 is the identity element wi...

(d) (i) Here, $5 + 5' = 5 + \frac{30}{5} = 5 + 6 = 30$

Similarly for each $x \in$ B, there e...

(ii) We have

$5 \cdot 5' = 5 \cdot \frac{30}{5} = 5 \cdot 6 = 1$

$10 \cdot 10' = 10 \cdot \frac{30}{10} = 10 \cdot 3 = 1$

Similarly for each $x \in$ B, there exi...

Thus B forms a Boolean algebra ...

and unitary operation '′'.

Now, by given definitions

$5 + 6$ = l.c.m. of $5 \& 6 = 30 \in B$

$5 \cdot 6$ = g.c.d. of $5 \& 6 = 1 \in B$

Same results hold for other elements also.

Thus 'B' is closed under '+' and '·'

(a) (i) We have

$x + y$ = l.c.m. of $x \& y$ = l.c.m. of $y \& x = y + x$

(ii) Here $x \cdot y$ = g.c.d. of $x \& y$ = g.c.d. of $y \& x = y \cdot x$

So, '+' and '·' are commutative.

(b) (i) We have

$5 + 3 \cdot 6 = 5 + 3 = 15$

$(5 + 3) \cdot (5 + 6) = 15 \cdot 30 = 15$

Thus, $5 + 3 \cdot 6 = (5 + 3) \cdot (5 + 6)$

Similarly result holds for other elements also.

i.e. $x + y \cdot z = (x + y) \cdot (x + z)$, $\forall x, y, z \in B$

(ii) We have

$3 \cdot (10 + 15) = 3 \cdot 30 = 3$

$(3 \cdot 10) + (3 \cdot 15) = 1 + 3 = 3$

$\therefore 3 \cdot (10 + 15) = (3 \cdot 10) + (3 \cdot 15)$

Similarly result holds for other elements also

i.e. $x \cdot (y + z) = x \cdot y + x \cdot z$, $\forall x, y, z \in B$

Thus, each of the binary operations distribute over other.

(c) (i) We have

$5 + 1 = 5$

i.e. $x + 1$ = l.c.m. of $x \& 1 = x$, $\forall x \in B$

So, 1 is the identity element under '+'.

(ii) Now, $x \cdot 30$ = g.c.d. of $x \& 30 = x$, $\forall x \in B$

So, 30 is the identity element with respect to multiplication.

(d) (i) Here, $5 + 5' = 5 + \frac{30}{5} = 5 + 6 = 30$

Similarly for each $x \in B$, there exists $x' \in B$ such that $x + x' = 30$

(ii) We have

$5 \cdot 5' = 5 \cdot \frac{30}{5} = 5 \cdot 6 = 1$

$10 \cdot 10' = 10 \cdot \frac{30}{10} = 10 \cdot 3 = 1$

Similarly for each $x \in B$, there exists $x' \in B$ such that $x \cdot x' = 1$

Thus B forms a Boolean algebra with respect to binary operations '+' and '·' and unitary operation ''.

12. **Prove that for any two elements a, b of a Boolean algebra $(B, +, \cdot)$, $a + b = 0 \Rightarrow a = 0, b = 0$.**

Solution Given $a + b = 0$, $\forall a, b \in B$

so, $a \cdot (a + b) = a \cdot 0$

$\Rightarrow (a + 0) \cdot (a + b) = 0$ [for each $a \in B$, $a \cdot 0 = 0$ & $a + 0 = a$]

$\Rightarrow a + (0 \cdot b) = 0$, each binary operation distributes over other

$\Rightarrow a + 0 = 0$

$\Rightarrow a = 0$

Hence $a + b = 0$ and $a = 0$

$\Rightarrow b = 0$

Thus, $a + b = 0 \Rightarrow a = 0$ and $b = 0$.

13. **In a Boolean algebra $(B, +, \cdot, ')$, prove that, $(a' + a \cdot b)' = (a \cdot b')$ $\forall a, b \in$**

[CH, 2011, 2

Solution We can write that

$(a + a \cdot b) + (a \cdot b') = a' + (a \cdot b + a \cdot b')$, each binary operations is associative

$= a' + a \cdot (b + b')$, each binary operation distributes over othe

$= a' + a \cdot 1$, for each $x \in B$, there exists $x' \in B$, such that $x + x' =$

$= a' + a = 1$

$\therefore (a' + a \cdot b) + (a \cdot b') = 1$

Again,

$(a' + a \cdot b) \cdot (a \cdot b')$

$= (a \cdot b') \cdot (a' + a \cdot b)$, each binary operation is commutative

$= (a \cdot b') \cdot a' + (a \cdot b') \cdot (a \cdot b)$, each binary operation is distributes over other

$= a \cdot (b' \cdot a') + a \cdot (b' \cdot a) \cdot b$, each binary operation is associative

$= a \cdot (a' \cdot b') + a \cdot (a \cdot b') \cdot b = (a \cdot a') \cdot b' + (a \cdot a) \cdot (b' \cdot b)$

$= 0 \cdot b' + a \cdot 0$ for each $x \in B$, there exist $x' \in B$, such that $x \cdot x' = 0$, $x \cdot x' =$

$= 0 + 0 = 0$

$\therefore (a' + a \cdot b) \cdot (a \cdot b') = 0$

Combining (1) and (2), we can conclude that $a \cdot b'$ is the complement of $(a' +$

$\therefore (a' + a \cdot b)' = a \cdot b'$

14. **Prove that $(a + b) \cdot (\bar{b} + c) + b \cdot (\bar{a} + \bar{c}) = a \cdot \bar{b} + a \cdot c + b$**

Solution LHS $= (a + b) \cdot (\bar{b} + c) + b \cdot (\bar{a} + \bar{c})$

$= a \cdot (\bar{b} + c) + b \cdot (\bar{b} + c) + b \cdot (\bar{a} + \bar{c})$

$= a \cdot (\bar{b} + c) + b \cdot (\bar{b} + \bar{a} + 1)$

$= a \cdot (\bar{b} + c) + b \cdot 1 = a \cdot \bar{b} + a \cdot c + b =$ RHS

15. **Prove that (i) $(a \cdot b) \cdot c = a \cdot (b \cdot c$**

Solution (i) Let $p = a \cdot (b \cdot c)$ and $q = ($

Then we have to show that $p = q$

(a) Since $a + p = a + q$,

therefore, $a + p = a + a \cdot (b \cdot c) = (a +$

$= a \cdot (a + c) = a \cdot a +$

$= a \cdot (a + c) = (a +$

$= a + (a \cdot b) \cdot c = a$

(b) To prove, $\bar{a} + p = \bar{a} + q$

let $\bar{a} + p = \bar{a} + a \cdot (b \cdot c) = (a + \bar{a})$

$= \bar{a} + b \cdot c = (\bar{a} + b) \cdot (\bar{a}$

$= 1(\bar{a} + b) \cdot (\bar{a} + c) = ($

$= (a + a \cdot b) \cdot (\bar{a} + c) =$

Let $p = q$, then from $(a)$and $(b$

$(a + p) \cdot (\bar{a} + p) = (a + q) \cdot (\bar{a}$

$\Rightarrow a \cdot \bar{a} + p = a \cdot \bar{a} + q$

$\Rightarrow 0 + p = 0 + q$

$\Rightarrow p = q$

Hence $a \cdot (b \cdot c) = (a \cdot b) \cdot c$

(ii) It is the dual of (i)

16. **If $x \in B$ then prove th**

(d) $x \cap 0 = 0$

Solution (a) $x \cup x = (x \cup x)$

$= (x \cup x) \cap (x \cup \bar{x}) = x \cup$

(b) $x \cap x = (x \cap x) \cup 0 = (x \cap$

$= x \cap (x \cup \bar{x}) = x \cap$

(c) $x \cup 1 = (x \cup 1) \cap 1 = (x \cup$

$= x \cup (1 \cap \bar{x}) = x \cup$

(d) $x \cap 0 = (x \cap 0) \cup 0 = (x \cap$

$= x \cap (0 \cup \bar{x}) = x \cap$

17. **If $x \in B$, then prove t**

Solution (a) $x \cup (x \cap y) =$

(b) Left as an exercise for t

15. **Prove that (i) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ (ii) $(a + b) + c = a + (b + c)$**

Solution (i) Let $p = a \cdot (b \cdot c)$ and $q = (a \cdot b) \cdot c$.

Then we have to show that $p = q$

(a) Since $a + p = a + q$,

therefore, $a + p = a + a \cdot (b \cdot c) = (a + a \cdot b) \cdot (a + c)$

$= a \cdot (a + c) = a \cdot a + a \cdot c = a + a \cdot c = a$

$= a \cdot (a + c) = (a + a \cdot b) \cdot (a + c)$

$= a + (a \cdot b) \cdot c = a + q$

(b) To prove, $\bar{a} + p = \bar{a} + q$

let $\bar{a} + p = \bar{a} + a \cdot (b \cdot c) = (a + \bar{a}) \cdot (a + b \cdot c) = 1 \cdot (\bar{a} + b \cdot c)$

$= \bar{a} + b \cdot c = (\bar{a} + b) \cdot (\bar{a} + c)$

$= 1(\bar{a} + b) \cdot (\bar{a} + c) = (\bar{a} + a) \cdot (\bar{a} + b)(\bar{a} + c)$

$= (a + a \cdot b) \cdot (\bar{a} + c) = \bar{a} + (a \cdot b \cdot c) = \bar{a} + q$

Let $p = q$, then from $(a)$and (b), we have

$(a + p) \cdot (\bar{a} + p) = (a + q) \cdot (\bar{a} + q)$

$\Rightarrow a \cdot \bar{a} + p = a \cdot \bar{a} + q$

$\Rightarrow 0 + p = 0 + q$

$\Rightarrow p = q$

Hence $a \cdot (b \cdot c) = (a \cdot b) \cdot c$

(ii) It is the dual of (i)

16. **If $x \in B$ then prove that (a) $x \cup x = x$; (b) $x \cap x = x$; (c) $x \cup 1 = 1$; (d) $x \cap 0 = 0$**

Solution (a) $x \cup x = (x \cup x) \cap 1$ [Since 1 is the identity element]

$= (x \cup x) \cap (x \cup \bar{x}) = x \cup (x \cap \bar{x}) = x \cup 0 = x$

(b) $x \cap x = (x \cap x) \cup 0 = (x \cap x) \cup (x \cap x)$

$= x \cap (x \cup \bar{x}) = x \cap 1 = x$

(c) $x \cup 1 = (x \cup 1) \cap 1 = (x \cup 1) \cap (x \cup \bar{x})$

$= x \cup (1 \cap \bar{x}) = x \cup \bar{x} = 1$

(d) $x \cap 0 = (x \cap 0) \cup 0 = (x \cap 0) \cup (x \cap \bar{x})$

$= x \cap (0 \cup \bar{x}) = x \cap x = 0$

17. **If $x \in B$,then prove that (a) $x \cup (x \cap y) = x$, and (b) $x \cap (x \cup y) = x$**

Solution (a) $x \cup (x \cap y) = (x \cap 1) \cup (x \cap y) = x \cap (1 \cup y) = x \cap 1 = x$

(b) Left as an exercise for the reader.

18. **If $(B, +, \cdot)$ is a Boolean algebra, then prove that the complement of any element $a \in B$ is unique.**

Solution Let $a \in B$ and suppose that its complement is not unique. If possible, suppose that $\bar{a}$ and $\bar{\bar{a}}$ are two complements of a such that

$a + \bar{a} = 1$ and $a \cdot \bar{a} = 0$

$a + \bar{\bar{a}} = 1$ and $a \cdot \bar{\bar{a}} = 0$

Now $\bar{a} = \bar{a} \cdot 1 = \bar{a} \cdot (a + \bar{\bar{a}})$

$= \bar{a} \cdot a + \bar{a} \cdot \bar{\bar{a}} = 0 + \bar{a} \cdot \bar{\bar{a}}$

$= a \cdot \bar{\bar{a}} + \bar{a} \cdot \bar{\bar{a}} = (a + \bar{a}) \cdot \bar{\bar{a}} = 1 \cdot \bar{\bar{a}} = \bar{\bar{a}}$

It follows that $\bar{a} = \bar{\bar{a}}$. Which contradicts the assumption that an element has two different complements. Hence every elements in Boolean algebra has a unique complement.

19. **In a certain town all cars marked P belong to the police. The cars are not fitted with the wireless unless they are marked P. The police have no blue cars. Prove by using the Boolean algebra that no blue cars are fitted with the wireless.**

Solution Let us denote letters p, q, r and s by the following statements

p = cars belongs to the police. q = cars are fitted with the wireless

r = Cars are marked P. s = cars are blue

Therefore, by given condition, we have $r \Rightarrow p$, $q \Rightarrow r$, $p \Rightarrow s$

But if $r \Rightarrow p$ and $p \Rightarrow \bar{s}$, then $r \Rightarrow \bar{s}$

Also if $q \Rightarrow r$ and $r \Rightarrow \bar{s}$, then $q \Rightarrow \bar{s}$

Or $s \cdot q \Rightarrow s \cdot \bar{s} = 0$

Hence it follows that no blue cars are fitted with the wireless

20. **Show that (a) $(x \cap y) \cup \overline{[(x \cup \bar{y}) \cap y]} = I$ and (b) $\overline{[(x \cdot y)(\bar{z} + x\bar{y})]} = \bar{x} + \bar{y} + z \cdot y$**

Solution (a) LHS $= (x \cap y) \cup \overline{[(x \cup \bar{y}) \cap y]}$

$= (x \cap y) \cup [\overline{(x \cup \bar{y})} \cup \bar{y}]$

$= (x \cap y) \cup [(\bar{x} \cap y) \cup \bar{y}]$

$= (x \cap y) \cup (\bar{x} \cup \bar{y}) = I = \text{RHS}$

(b) LHS $= \overline{[(x \cdot y)(\bar{z} + x \cdot \bar{y})]} = \overline{(x \cdot y)} + \overline{(\bar{z} + x \cdot \bar{y})} = (\bar{x} + \bar{y}) + [z \cdot (\bar{x} + y)]$

$= \bar{x} + \bar{y} + z\bar{x} + z \cdot y = \bar{x} + \bar{x} \cdot z + \bar{y} + y \cdot z = \bar{x}(1 + z) + \bar{y} + y \cdot z$

$= \bar{x} \cdot 1 + \bar{y} + y \cdot z = \bar{x} + \bar{y} + y \cdot z = \text{RHS}$

The result (a) can be verified by the t

Let $a = x \cap y$, $b = (x \cup \bar{y}) \cap y$.

x	y	$a = x \cap y$	$\bar{y}$	$x \cup$
1	1	1	0	1
1	0	0	1	1
0	1	0	0	0
0	0	0	1	1

21. **Prove that $\overline{[(\bar{x} \cap \bar{y}) \cup z] \cap (x \cup z)}$**

Solution LHS $= \overline{[(\bar{x} \cap \bar{y}) \cup z] \cap (x \cup z)}$

$= [\overline{(\bar{x} \cap \bar{y}) \cup z}] \cup [\bar{x} \cap \bar{z}] = (\bar{x} \cap \bar{y} \cap \bar{z}$

$= \bar{x} \cap \bar{z} = \text{RHS}$

22. **Simplify $\overline{[(x \cup \bar{y}) \cap (x \cap \bar{y} \cap \bar{z})]}$**

Solution $\overline{[(x \cup \bar{y}) \cap (x \cap \bar{y} \cap \bar{z})]} = \overline{(x \cup \bar{y}}$

$= (\bar{x} \cap y) \cup (\bar{x} \cup \bar{\bar{y}} \cup \bar{\bar{z}}) = (\bar{x} \cap y)$

23. **Simplify the Boolean expres**

Solution We have,

$x + \bar{x} \cdot (x + y) + y \cdot z = x + (\bar{x} \cdot x + \bar{x} \cdot y$

$= x + (0 + \bar{x} \cdot y) + y \cdot z = x + \bar{x} \cdot y +$

$= (x + \bar{x}) \cdot (x + y) + y \cdot z = 1 \cdot (x + y)$

$= (x + y) + y \cdot z = x + y \cdot (1 + z) = x$

24. **Using properties of Boolean algebra, if $x + y = 1$ and $x \cdot y$**

Solution We have,

$\bar{x} = \bar{x} + 0 = \bar{x} + x \cdot y = (\bar{x} + x) \cdot (\bar{x} +$

$= 1 \cdot (\bar{x} + y) = (x + y) \cdot (\bar{x} + y)$

$= (y + x) \cdot (y + \bar{x}) = y + (x \cdot \bar{x}) = y$

25. **Simplify the Boolean expre**

Solution We have,

$(x + y) \cdot (x + z) \cdot \overline{(\bar{x} \cdot y)}$

$= (x + y) \cdot (x + z) \cdot (x + \bar{y}) = x + (y$

$= x + \bar{y} \cdot (y \cdot z) = x + (\bar{y} \cdot y) \cdot z = x$

The result (a) can be verified by the following truth table.
Let $a = x \cap y$, $b = (x \cup \bar{y}) \cap y$.

x	y	$a = x \cap y$	$\bar{y}$	$x \cup \bar{y}$	$b = (x \cup \bar{y}) \cap y$	$\bar{b}$	$a \cup \bar{b}$
1	1	1	0	1	1	0	1
1	0	0	1	1	0	1	1
0	1	0	0	0	1	1	1
0	0	0	1	1	0	1	1

21. **Prove that** $\overline{[\{(\bar{x} \cap \bar{y}) \cup z\} \cap (x \cup z)]} = \bar{x} \cap \bar{z}$

Solution LHS $= \overline{[\{(\bar{x} \cap \bar{y}) \cup z\} \cap (x \cup z)]} = [\overline{\{(\bar{x} \cap \bar{y}) \cup z\}} \cup \overline{(x \cup z)}]$

$= [\{(x \cup y) \cup z\} \cup \{\bar{x} \cap \bar{z}\} = (\bar{x} \cap \bar{y} \cap \bar{z}) \cup (\bar{x} \cap \bar{z})$

$= \bar{x} \cap \bar{z} =$ RHS

22. **Simplify** $\overline{[(x \cup \bar{y}) \cap (x \cap \bar{y} \cap \bar{z})]}$

Solution $\overline{[(x \cup \bar{y}) \cap (x \cap \bar{y} \cap \bar{z})]} = \overline{(x \cup \bar{y})} \cup \overline{(x \cap \bar{y} \cap \bar{z})}$

$= (\bar{x} \cap y) \cup (\bar{x} \cup \bar{\bar{y}} \cup \bar{\bar{z}}) = (\bar{x} \cap y) \cup (\bar{x} \cup y \cup z)$

23. **Simplify the Boolean expression :** $x + \bar{x} \cdot (x + y) + y \cdot z$

Solution We have,

$x + \bar{x} \cdot (x + y) + y \cdot z = x + (\bar{x} \cdot x + \bar{x} \cdot y) + y \cdot z$

$= x + (0 + \bar{x} \cdot y) + y \cdot z = x + \bar{x} \cdot y + y \cdot z$

$= (x + \bar{x}) \cdot (x + y) + y \cdot z = 1 \cdot (x + y) + y \cdot z$

$= (x + y) + y \cdot z = x + y \cdot (1 + z) = x + y \cdot 1 = x + y$

24. **Using properties of Boolean algebra, prove that for each x in a Boolean algebra, if $x + y = 1$ and $x \cdot y = 0$ then $y = \bar{x}$**

Solution We have,

$\bar{x} = \bar{x} + 0 = \bar{x} + x \cdot y = (\bar{x} + x) \cdot (\bar{x} + y)$

$= 1 \cdot (\bar{x} + y) = (x + y) \cdot (\bar{x} + y)$

$= (y + x) \cdot (y + \bar{x}) = y + (x \cdot \bar{x}) = y + 0 = y$

25. **Simplify the Boolean expression :** $(x + y) \cdot (x + z) \cdot \overline{(\bar{x} \cdot y)}$

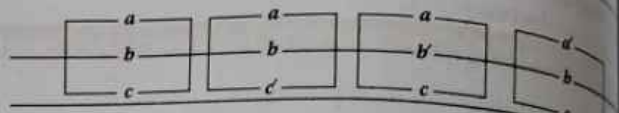
Solution We have,

$(x + y) \cdot (x + z) \cdot \overline{(\bar{x} \cdot y)}$

$= (x + y) \cdot (x + z) \cdot (x + \bar{y}) = x + (y \cdot z) \cdot \bar{y}$

$= x + \bar{y} \cdot (y \cdot z) = x + (\bar{y} \cdot y) \cdot z = x + 0 \cdot z = x + 0 = x$

26. **Simplify the following switching circuit,**



Solution Switching function corresponding to given circuit,

$f = (a + b + c)(a + b + c')(a + b' + c)(a' + b + c)$

$= \{(a + b) + (c \cdot c')\}[\{(a + b') \cdot (a' + b)\} + c]$,

each binary operation distribution over

$= \{(a + b) + 0\}[\{(a + b') \cdot a' + (a + b') \cdot b\} + c]$

$= \{(a + b) + 0\}[\{aa' + b'a + ab + bb'\} + c]$,

each binary operation distribution over

$= (a + b)\{(ab + a'b') + c\}$

$= (a + b)(ab) + (a + b)a'b' + (a + b) \cdot c$

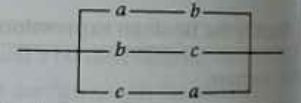
$= aab + bab + aa'b' + ba'b' + ac + bc$

$= ab + ab + 0 + 0 + ac + bc$

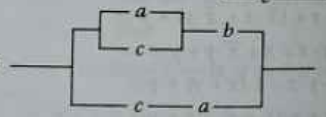
$= ab + bc + ca$

$= (a + c)b + ca$

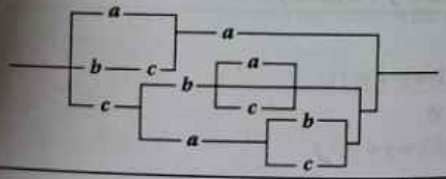
Simplified circuit can be expressed as



or, equivalent circuit as



27. **Find the simplest switching circuit equivalent to the following switching circuit,**



Solution Corresponding switching function to the given circuit,

$f = (a + bc)a + c[b(a + c) + a(b + c)]$

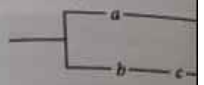
$= (aa + abc) + c[ab + bc + ab + ac]$

$= (a + abc) + abc + bc + abc + ac$

$= a + abc + bc + ac = a(1 + bc) + bc + ac$

$= a(1 + c) + bc = a + bc$

Hence, the simplified circuit is,



28. **Draw a switching circuit which realiz**
$x + y[z + x'(y' + z')]$.

Solution Given expression is $x + y[z + x'(y'$

Required switching circuit is

29. **Write a switching function for (whenever possible) and draw the s**

Solution The switching function for the

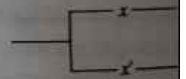
$f = (xy + x'y + xy')(x' + y')$

$= \{x(y + y') + x'y\}(x' + y') = (x + x'y)(x' + y$

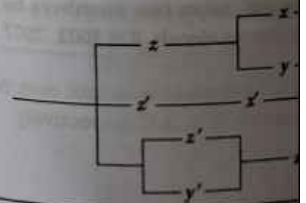
$= x(x' + y') + x'y(x' + y') = xx' + xy' + x'yx'$

$= xy' + x'y$

So, simplified circuit is



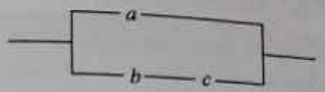
30. **Find simpler equivalent switching**



Solution The switching function is

$f = z(x + y) + z'x' + z'(z' + y')$

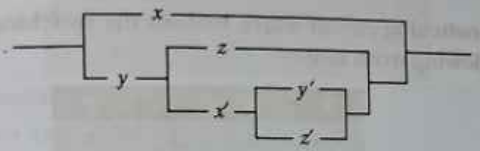
Hence, the simplified circuit is,



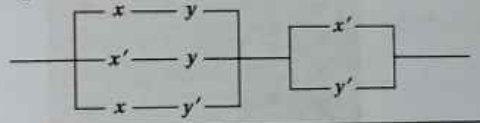
28. **Draw a switching circuit which realizes the expression $x + y[z + x'(y' + z')]$.** [CH, 2000]

Solution Given expression is $x + y[z + x'(y' + z')]$.

Required switching circuit is



29. **Write a switching function for the following circuit, simplify it (whenever possible) and draw the simplified circuit,** [CH, 2019]



Solution The switching function for the given circuit is

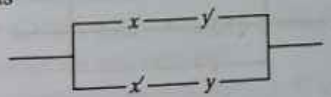
$f = (xy + x'y + xy')(x' + y')$

$= [x(y + y') + x'y](x' + y') = (x + x'y)(x' + y')$

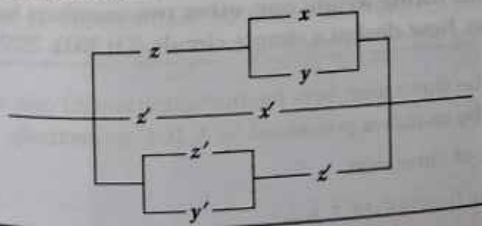
$= x(x' + y') + x'y(x' + y') = xx' + xy' + x'yx' + x'yy'$

$= xy' + x'y$

So, simplified circuit is



30. **Find simpler equivalent switching circuit for the following circuit**



[CH, 2001, 2013]

Solution The switching function is

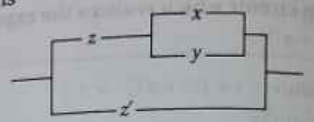
$f = z(x + y) + z'x' + z'(z' + y')$

$= zx + zy + x'z' + z' + z'y' = zx + zy + (x' + 1)z' + z'y'$

$= zx + zy + z'(1 + y') = zx + zy + z'$

$= z(x + y) + z'$

So, simplified circuit is



31. **Find the switching circuit which realises the switching function by the following truth table :**

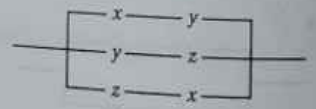
x	y	z	f
1	1	1	1
1	1	0	1
1	0	1	1
1	0	0	0
0	1	1	1
0	1	0	0
0	0	1	0
0	0	0	0

[CH 2

Solution The Switching function corresponding to given truth table is

$f = xyz + xyz' + xy'z + x'yz = xy + yz + zx$

So, the switching circuit is



32. **A committee of three persons decides proposal by calculating votes one member has voting weight one, other two members having weight two, two. Now, design a simple circuit.** [CH 2003, 2007, 2012

Solution Let A, B, C be three members having voting weight one, two and respectively and x, y, z be switches possessed by A, B, C respectively.

Let v be the number of votes cast.

Let f be the switching function of x, y, z.

We assume that, light coil glow when $f(x, y, z) = 1$

and $f(x, y, z) = 1$ when $v \geq 3$.

So, we construct the following truth

x	y
0	0
0	0
0	1
0	1
1	0
1	0
1	1
1	1

Thus required switching function

$= x'yz + xy'z + xyz' + xyz$

$= (xyz + x'yz) + (xy'z + xyz) + ($

$= xy + yz + zx = x(y + z) + yz$

So, required switching circuit:

33. **A committee consists o treasurer. A proposal is a votes or vote of the presi approves the proposal b switching circuit controll pass if and only if a prop**

Solution Let P, V, S, T be the president, secretary and treasurer

We have to construct a Boolea will glow if $f(P, V, S, T) = 1$.

So, we construct the following t

P	
0	
0	
0	
0	
0	
0	
0	

So, we construct the following truth table based on the given problem as follows

x	y	z	v	f
0	0	0	0	0
0	0	1	2	0
0	1	0	2	0
0	1	1	4	1
1	0	0	1	0
1	0	1	3	1
1	1	0	3	1
1	1	1	5	1

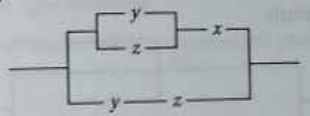
Thus required switching function,

$= x'yz + xy'z + xyz' + xyz$

$= (xyz + x'yz) + (xy'z + xyz) + (xyz' + xyz)$

$= xy + yz + zx = x(y + z) + yz$

So, required switching circuit:



33. **A committee consists of president, vice-president, secretary and treasurer. A proposal is accepted if and only if it receives a majority of votes or vote of the president plus one other member. Each member approves the proposal by pressing a bottom to their seats. Design a switching circuit controlled by the bottom which allow the current to pass if and only if a proposal is accepted.** [CH, 2004]

Solution Let P, V, S, T be the four switches possessed by president, vice-president, secretary and treasurer respectively.

We have to construct a Boolean function $f(P, V, S, T)$ and we assume that light will glow if $f(P, V, S, T) = 1$.

So, we construct the following truth table based on the given problem as follows

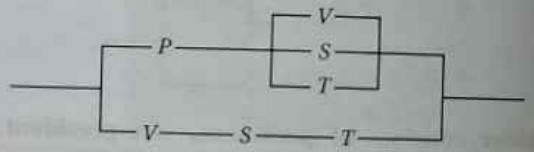
P	V	S	T	f
0	0	0	0	0
0	0	0	1	0
0	0	1	0	0
0	0	1	1	0
0	1	0	0	0
0	1	0	1	0
0	1	1	0	0

P	V	S	T	f
0	1	1	1	1
1	0	0	0	0
1	0	0	1	1
1	0	1	0	1
1	0	1	1	1
1	1	0	0	1
1	1	0	1	1
1	1	1	0	1
1	1	1	1	1

Thus required switching function is

$$f = P'VST + PV'S'T + PV'ST' + PV'ST + PVS'T' + PVS'T + PVST'$$
$$= (P' + P)VST + PV'T(S' + S) + PST'(V' + V) + PVS'(T' + T)$$
$$= VST + PV'T + PST' + PVS' = P(V'T + VS' + ST') + VST'$$
$$= P(V + S + T) + VST$$

Required switching circuit,



34. **Show that current will flow through the network given by the function $[xy(x'y + xy')]'$ irrespective of the switches x and / or y are closed or open.**

Solution Let $f = [xy(x'y + xy')]'$

$= (xy)' + (x'y + xy')'$, by using De Morgan's law

$= (x' + y') + (x + y')(x' + y)$, by using De Morgan's law

$= x' + y' + (x + y')x' + (x + y')y$, each binary operation distributes over

$= x' + y' + xx' + y'x' + xy + y'y$

$= x' + y' + x'y' + xy$, $[\because a \cdot a' = 0$, for all $a]$

$= x' + xy + y'(I + x') = (x' + y') + xy = (xy)' + xy = I$.

This shows that, current will always flow through the network and it does not depend on state of x and y/x or y.

35. **Suppose S is a set of propositional formulas and F be a propositional formula. When is F said to be a logical consequence of the formulas S?**

Solution We can justify our statement if we consider that S is a set of Propositional formula $p, q, r, \ldots$ and F is a propositional formula and $p, q, r, \ldots$

true then F is true. In this case F can express as $p, q, r, \ldots, \vdash F$ (thi
We justify this as $(p \wedge q \wedge r \wedge \ldots)$
The argument is a valid argu
premises are true. The argument
group of propositions $p, q, r, \ldots$ i
$(p \wedge q \wedge r \wedge \ldots) \rightarrow F$ is a tautol

36. **Determine whether the answer. If the formula**

Solution We have $(p \vee (p \wedge q$
Hence, if $(p \vee (p \wedge q))$ is true

A Short and Long Answer Type

1. Determine which of the or contradictions :
   (i) $(p \rightarrow \sim p) \rightarrow \sim p$
   (ii) $p \rightarrow (p \vee q)$
   (iii) $(\sim q \rightarrow p) \wedge q$
   (iv) $(p \vee q) \wedge (\sim p \vee r) \rightarrow (q \vee$
   (v) $((p \rightarrow (q \rightarrow r)) \rightarrow ((p \rightarrow$
   (vi) $p \wedge (q \wedge r) \leftrightarrow (p \wedge q) \wedge$
   (vii) $(p \vee q) \wedge (\sim q) \wedge (\sim p)$
2. Show that the following truth tables :
   (i) $p \leftrightarrow (\sim p \vee \sim q)$
   (ii) $(\sim p \vee \sim q) \rightarrow q$
   (iii) $(p \rightarrow q \wedge r) \rightarrow (p \rightarrow q)$
3. Prove that the following $[(p \vee q) \wedge (p \rightarrow r) \wedge (p \rightarrow$
4. Prove the following equiv
   (i) $\sim(p \rightarrow q) \equiv p \wedge \sim q$
   (ii) $(p \wedge q) \vee (p \wedge \sim q) \equiv p$
   (iii) $(p \vee q) \wedge (p \vee \sim q) \equiv p$
   (iv) $\sim(p \leftrightarrow q) \equiv (p \wedge \sim q) \vee (\sim$
   (v) $(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow$

true then F is true. In this case F is a logical consequence of S. Mathematically, we can express as $p, q, r, \ldots, \vdash F$ (this means $p, q, r, \ldots$ imply F or $p, q, r, \ldots$ prove F).

We justify this as $(p \wedge q \wedge r \wedge \ldots) \rightarrow F$. Thus, F is a logical consequence of $p, q, r, \ldots$.

The argument is a valid argument if the conclusion is true whenever all the premises are true. The argument is also valid if and only if the conjunctions of the group of propositions $p, q, r, \ldots$ implies conclusion F is a tautology i.e., $(p \wedge q \wedge r \wedge \ldots) \rightarrow F$ is a tautology.

36. Determine whether the following statement is true or false. Justify your answer. If the formula $(p \vee (p \wedge q))$ is true, then p is true. [CU, 2021]

Solution We have $(p \vee (p \wedge q)) \equiv p$ [by absorption law]

Hence, if $(p \vee (p \wedge q))$ is true then p is true

Exercise

Short and Long Answer Type Questions

1. Determine which of the following compound propositions are tautologies or contradictions :
   (i) $(p \rightarrow \neg p) \rightarrow \neg p$
   (ii) $p \rightarrow (p \vee q)$
   (iii) $(\neg q \rightarrow p) \wedge q$
   (iv) $(p \vee q) \wedge (\neg p \vee r) \rightarrow (q \vee r)$
   (v) $((p \rightarrow (q \rightarrow r)) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r))$
   (vi) $p \wedge (q \wedge r) \leftrightarrow (p \wedge q) \wedge r$
   (vii) $(p \vee q) \wedge (\neg q) \wedge (\neg p)$
2. Show that the following compound propositions are contingency using truth tables :
   (i) $p \leftrightarrow (\neg p \vee \neg q)$
   (ii) $(\neg p \vee \neg q) \rightarrow q$
   (iii) $(p \rightarrow q \wedge r) \rightarrow (p \rightarrow q)$
3. Prove that the following proposition is Tautology (without truth table) [VU, 2019]
   $[(p \vee q) \wedge (p \rightarrow r) \wedge (p \rightarrow r)] \rightarrow r$
4. Prove the following equivalences:
   (i) $\neg(p \rightarrow q) \equiv p \wedge \neg q$
   (ii) $(p \wedge q) \vee (p \wedge \neg q) \equiv p$
   (iii) $(p \vee q) \wedge (p \vee \neg q) \equiv p$
   (iv) $\neg(p \leftrightarrow q) \equiv (p \wedge \neg q) \vee (\neg p \wedge q)$
   (v) $(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$

(vi) $p \wedge ((\sim q \vee r) \wedge (\sim r \vee q)) \equiv p \wedge (q \leftrightarrow r)$

(vii) $(\sim p \wedge q) \rightarrow (\sim p \vee (\sim p \vee q)) \equiv \sim p \vee q$ without using truth table

5. Determine the dual of the following statements

(i) $p \wedge (q \vee r)$

(ii) $\sim p \wedge \sim q$

(iii) $(p \wedge \sim q) \wedge (\sim p \wedge q)$

(iv) $((\sim p \vee q) \wedge (q \wedge \sim r)) \vee (p \vee \mathrm{T})$, where T is true

6. State which rule of inference is used in the argument

(i) Kangaroos live in Australia and are marsupials. Therefore, Kangaroos are marsupials.

(ii) Covid 19 is a Pandamic and originated from china. Therefore, Covid 19 originated from China

(iii) Hema is a very good dancer. If Hema is a very good dancer, then she can work as a dance trainer. Therefore, Hema can work as a dance trainer.

(iv) Aloke will work at a computer company this winter. Therefore, this winter Aloke will work at a computer company or he will be a beach bum.

(v) If I work all night on my lessons, then I can answer all the questions. If I answer all the questions, I will have better knowledge. Therefore, if I work all night on my lessons, then I will have better knowledge

(vi) Either today I will go to cinema or I will go to shopping mall. I will not go to cinema today, therefore I will go to shopping mall.

7. Prove the validity of the following argument using truth tables

(i) If today is Sunday, then yesterday was Saturday. Yesterday was Saturday. Therefore, today is Sunday.

(ii) If a man is married then he is unhappy. If a man is unhappy then he dies young. Therefore, married man dies young.

(iii) Every living thing is a plant or an animal. My cat is alive and it is not a plant. All animals have hearts. Therefore my cat has a heart.

(iv) All integers are rational numbers. Some integers are powers of 2. Therefore, some rational numbers are powers of 2.

(v) If I like Propositional logic, then I will study. Either I study or I get low marks. Therefore, if I get less marks, I do not like Propositional logic.

(vi) If the wages increase, then there will be inflation. The cost of living will decrease if there is no inflation. Therefore, if wages will increase then the cost of living will not decrease.

8. Prove the validity of the following arguments without using truth table

(i) $p, q \vdash (p \vee r) \wedge q$

(ii) $p \rightarrow q, p \wedge r \vdash q$

(iii) $(p \rightarrow q) \wedge (r \rightarrow s), (p \vee r) \wedge (q \vee r) \vdash q \vee s$

(iv) $(p \wedge q) \vee (r \rightarrow s), t \rightarrow r, \sim (p$

(v) $p \rightarrow (q \vee r), (s \wedge t) \rightarrow q, (q$

(vi) $p \vee (q \rightarrow p), \sim p \wedge r \vdash \sim q$

9. Determine whether the given not meet Sajal last night, th lying. If Sajal was not the mu and the murder took place midnight, then either Sajal w Sajal was the murderer.

10. Prove the validity of the foll

(i) Either I will get good mar will go to Europe. I got go

(ii) If the market is free, ther then there are price cont the market is free.

(iii) One student in Compute Everyone who knows how job. Therefore, someone i

(iv) Everyone who eats some Therefore, Jaya does not

(v) A student in this class passed the first examina examination has not read

(vi) No man is an island. Man a man.

11. Using Method of Mathema

(i) Prove that if n^2 is odd th

(ii) Prove that $\sqrt{2}$ is not a ra

(iii) Prove that for every posi

(iv) Prove that the sum of fir

(v) Prove that $1^3 + 2^3 + 3^3 +$

(vi) Prove that for all positive

$$\frac{1^2}{1\cdot 3} + \frac{2^2}{3\cdot 5} + \frac{3^2}{5\cdot 7} + \ldots \frac{n}{(2n-1)}$$

(vii) Prove that for all positive

$1\cdot 2 + 2\cdot 3 + 3\cdot 4 + \ldots + n(n$

(viii) Prove that $2^{2n} - 1$ is divis

(iv) $(p \wedge q) \vee (r \to s),\ t \to r,\ \sim(p \wedge q) \vdash t \to s$

(v) $p \to (q \vee r),\ (s \wedge t) \to q,\ (q \vee r) \to (s \wedge t) \vdash p \to q$

(vi) $p \vee (q \to p),\ \sim p \wedge r \vdash \sim q$

9. Determine whether the given arguments are logically correct: If Samar did not meet Sajal last night, then either Sajal was the murderer or Samar is lying. If Sajal was not the murder, then Samar did not meet Sajal last night and the murder took place after midnight. If the murder took place after midnight, then either Sajal was the murderer or Samar is not lying. Hence Sajal was the murderer. [CU, 2021]

10. Prove the validity of the following argument without using truth tables

(i) Either I will get good marks or I will not graduate. If I did not graduate, I will go to Europe. I got good marks. Therefore, I will not go to Europe.

(ii) If the market is free, then there is no inflation. If there is no inflation, then there are price controls. Since, there are price controls; therefore, the market is free.

(iii) One student in Computer class knows how to write programs in C++. Everyone who knows how to write programs in C++ can get a high-paid job. Therefore, someone in this class can get a high-paid job.

(iv) Everyone who eats some vegetables daily are healthy. Jaya is not healthy. Therefore, Jaya does not eat vegetables daily.

(v) A student in this class has not read the book. Everyone in this class passed the first examination. Therefore someone who passed the first examination has not read the book.

(vi) No man is an island. Manhattan is an island. Therefore, Manhattan is not a man.

11. Using Method of Mathematical Induction

(i) Prove that if n^2 is odd then n is also odd where n is an integer.

(ii) Prove that $\sqrt{2}$ is not a rational number.

(iii) Prove that for every positive integer n, $n^2 + n$ is always even integer.

(iv) Prove that the sum of first n odd integers is n^2 for any positive integer n.

(v) Prove that $1^3 + 2^3 + 3^3 + \ldots + n^3 = \left[\frac{n(n+1)}{2}\right]^2,\ n \in \mathbb{N}$.

(vi) Prove that for all positive integer n

$$\frac{1^2}{1\cdot3} + \frac{2^2}{3\cdot5} + \frac{3^2}{5\cdot7} + \ldots \frac{n^2}{(2n-1)(2n+1)} = \frac{n(n+1)}{2(2n+1)}$$

(vii) Prove that for all positive integer n

$$1\cdot2 + 2\cdot3 + 3\cdot4 + \ldots + n(n+1) = \frac{n(n+1)(n+2)}{3}$$

(viii) Prove that $2^{2n} - 1$ is divisible by 3 for every positive integer 'n'.

(ix) Prove that for all positive integer 'n', $n^2 < 2^n$, $\forall\ n \geq 5$

(x) Prove that $2 \cdot 7^n + 3 \cdot 5^n - 5$ is divisible by 24, for all $n \in \mathbb{N}$.

(xi) Prove that ${}^nC_1 + {}^nC_2 + \ldots + {}^nC_n = 2^n - 1$, for $n > 1$.

(xii) Prove that $\sin x + \sin 2x + \ldots + \sin nx = \dfrac{\sin\left(\frac{nx}{2}\right)\sin\left(\frac{n+1}{2}\right)x}{\sin\left(\frac{x}{2}\right)}$

(xiii) Prove that $2^{2n+1} - 9n^2 + 3n - 2$ is divisible by 54.

(xiv) Prove that if $a_1, a_2, \ldots a_n$ are n positive proper fraction (n be natural number) then
$(1 - a_1)(1 - a_2) \ldots (1 - a_n) \geq 1 - (a_1 + a_2 + \ldots + a_n)$.

(xv) For any finite set A, prove that $|P(A)| = 2^A$ or prove that there are 2^n in a set of elements.

(xvi) Prove that $(3 + \sqrt{5})^n + (3 - \sqrt{5})^n$ is divisible by 2^n, $\forall n \in \mathbb{N}$.

12. Considering Fibonacci sequence $\{f_1, f_2, f_3, \ldots\}$ where $f_1 = 1$, $f_2 =$ $f_n = f_{n-1} + f_{n-2}$ for $n \geq 3$, prove that $f_n \geq u^{n-2}$, where $u = \frac{1+\sqrt{5}}{2}$ and

13. Give a direct proof of the following statements

(i) Sum of two odd integers is even

(ii) Sum of two rational numbers is rational

(iii) Square of an even number is an even number

(iv) If n is an integer and $n^2 + 5$ is odd, then n is even

(v) Product of two rational numbers is rational

14. Give an indirect proof of the following statements

(i) If $x, y \in \mathbb{Z}$ (set of integers) such that xy is odd then both x and y are

(ii) If x and y are two integers such that x is even and y is odd then xy

(iii) If $x^2 + 4x = 0$, $x \in \mathbb{R}$ (set of Reals) then $x = 0$

(iv) For all real numbers x and y, if $x + y \geq 2$ then either $x \geq 1$ or $y \geq 1$

(v) If n is a positive integer, then n is odd if and only if $5n + 6$ is odd

(vi) For $x, y \in \mathbb{Z}$ (set of integers), $x^2 = y^2$ if and only if $x = y$ or $x = -y$

15. Prove that $\sqrt{2}$ is irrational by giving a proof of contradiction [MU

16. Find an equivalent expression for the expression $(p \to q \wedge r) \vee (r \leftrightarrow$ contains minimal complete set of connectives.

17. Find the DNF of the following statements :

(i) $\sim(p \to (q \wedge r))$ (ii) $(\sim p \to r) \wedge (p \leftrightarrow q)$

(iii) $p \wedge (p \to q)$ (iv) $q \wedge (p \vee \sim q)$

18. Find the CNF of the following statements:

(i) $\sim(p \vee q) \leftrightarrow (p \wedge q)$ (ii) $(\sim p \to r) \wedge (q \leftrightarrow p)$

(iii) $\sim((p \vee \sim q) \wedge \sim r)$ (iv) $q \vee (p \wedge q) \wedge \sim((p \vee r) \wedge q)$

19. Find the PDNF of the following state

(i) $p \wedge (p \to q))$ (ii)

(iii) $q \wedge (p \vee \sim q)$ (iv)

20. Find the PCNF of the following state

(i) $p \leftrightarrow q$ (ii)

(iii) $\sim(p \vee q)) \vee (p \wedge q)$ (iv)

(v) $p \to (p \wedge (q \to p))$

21. Find the PDNF and PCNF of the fo

(i) $p \to (p \wedge (q \to p))$ (ii)

(iii) $(p \wedge q) \vee (p \wedge r) \vee (q \wedge r)$ (iv)

22. Using truth table, find the principa principal conjunctive normal form

(i) $p \to (p \wedge (q \to p))$ (ii

(iii) $(p \wedge q) \vee (p \wedge r) \vee (q \wedge r)$ (iv

23. Let B be the set of all positive divi

$a + b = l.c.m$ of a & b; $a \cdot b = g.c.d$ o

Examine whether $(B, +, \cdot, ')$ is a Bo

24. Prove that set of all positive divis respect to two binary operations follows :

$a + b = l.c.m$ of a & b; $a \cdot b =$ g.c.d o

25. Let B be the set of all positive div

$a + b =$ l.c.m. of a & b; $a \cdot b =$ g.c.d

Examine whether $(B, +, \cdot, ')$ is a Bo

26. Prove that the complement of an

27. Let $(B, +, \cdot, ')$ be a Boolean algebr $\Rightarrow a = 0$, $b = 0$. Hence show that

28. Express the Boolean expression (in DNF. Also find CNF.

29. Write the following Boolean expr and DNF.

30. Draw the circuit diagram corresp

(i) $(x \wedge y) \vee (y \wedge z) \vee (z \wedge x)$ (ii) y

(iii) $z \vee (x \vee y) \wedge z'$ (iv) $x \wedge (y \wedge (z$

(v) $(x \wedge y \wedge z) \vee (\sim x \wedge (y \vee z))$

19. Find the PDNF of the following statements:
(i) $p \wedge (p \rightarrow q)$ (ii) $\sim(p \vee q) \leftrightarrow p \wedge q$
(iii) $q \wedge (p \vee \sim q)$ (iv) $(q \rightarrow p) \wedge (\sim p \wedge q)$

20. Find the PCNF of the following statements :
(i) $p \leftrightarrow q$ (ii) $(p \vee q) \rightarrow (p \wedge q)$
(iii) $\sim(p \vee q)) \vee (p \wedge q)$ (iv) $(q \rightarrow p) \wedge (\sim p \wedge q)$
(v) $p \rightarrow (p \wedge (q \rightarrow p))$

21. Find the PDNF and PCNF of the following statements using truth table:
(i) $p \rightarrow (p \wedge (q \rightarrow p))$ (ii) $(q \rightarrow p) \wedge (\sim p \wedge q)$
(iii) $(p \wedge q) \vee (p \wedge r) \vee (q \wedge r)$ (iv) $(p \wedge q) \vee (\sim p \wedge q) \vee (q \wedge r)$

22. Using truth table, find the principal disjunctive normal form (PDNF) and principal conjunctive normal form (PCNF) of the following statements :
(i) $p \rightarrow (p \wedge (q \rightarrow p))$ (ii) $(q \rightarrow p) \wedge (\sim p \wedge q)$
(iii) $(p \wedge q) \vee (p \wedge r) \vee (q \wedge r)$ (iv) $(p \wedge q) \vee (\sim p \wedge q) \vee (q \wedge r)$

23. Let B be the set of all positive divisor of 48 and let us define
$a + b = l.c.m$ of a & b; $a \cdot b = g.c.d$ of a & b; $a' = \frac{48}{a}$
Examine whether $(B, +, \cdot, ')$ is a Boolean algebra or not.

24. Prove that set of all positive divisors of 70 forms a Boolean algebra with respect to two binary operations '+', '·' and an unitary operation ''' as follows :
$a + b = l.c.m$ of a & b; $a \cdot b =$ g.c.d of a & b; $a' = \frac{70}{a}$

25. Let B be the set of all positive divisor of 72. Let us define
$a + b =$ l.c.m. of a & b; $a \cdot b =$ g.c.d of a & b; $a' = \frac{72}{a}$
Examine whether $(B, +, \cdot, ')$ is a Boolean algebra is or not.

26. Prove that the complement of an element in a Boolean algebra is unique. [CH, 2013]

27. Let $(B, +, \cdot, ')$ be a Boolean algebra and $a, b \in B$. Then prove that $a + b = 0 \Rightarrow a = 0, b = 0$. Hence show that $a \cdot b' + a' \cdot b = 0 \Rightarrow a = b$. [CH, 2019]

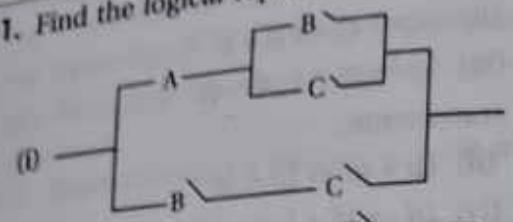
28. Express the Boolean expression $(x + y + z)(xy + x'z)'$ in the variables x, y, z in DNF. Also find CNF. [CH, 2001, 2006, 2012, 2016]

29. Write the following Boolean expression $xyz + (x + y)(x + z)$ in both CNF and DNF.

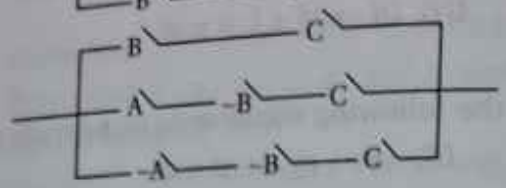
30. Draw the circuit diagram corresponding to the following functions:
(i) $(x \wedge y) \vee (y \wedge z) \vee (z \wedge x)$ (ii) $y \wedge (z \wedge \sim x \wedge (\sim y \vee \sim z')$
(iii) $z \vee (x \vee y) \wedge z'$ (iv) $x \wedge (y \wedge (z \vee w) \vee z \wedge (u \vee v))$
(v) $(x \wedge y \wedge z) \vee (\sim x \wedge (y \vee z))$

31. Find the logical representation for the following circuits :

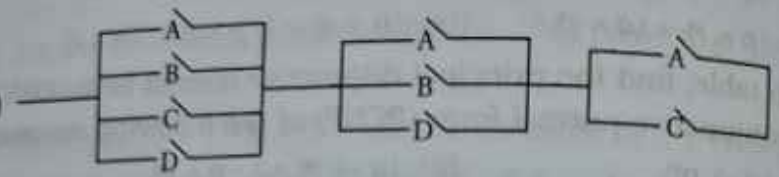
(i)



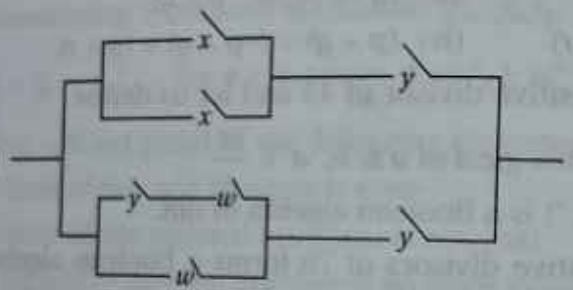
(ii)



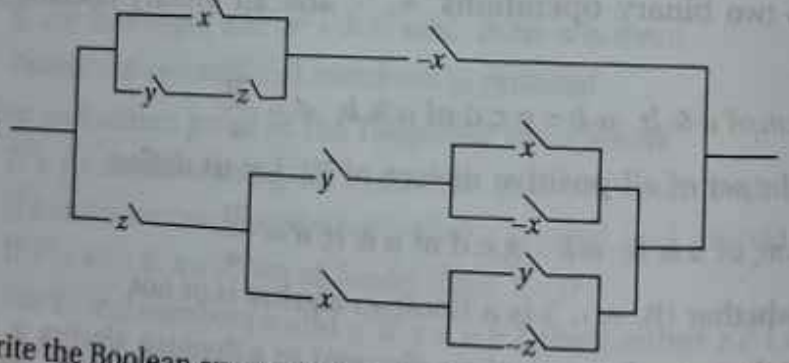
(iii)



(iv)

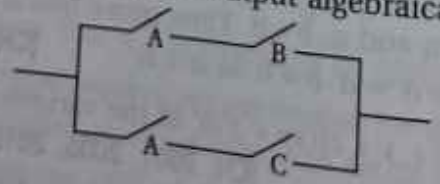


(v)

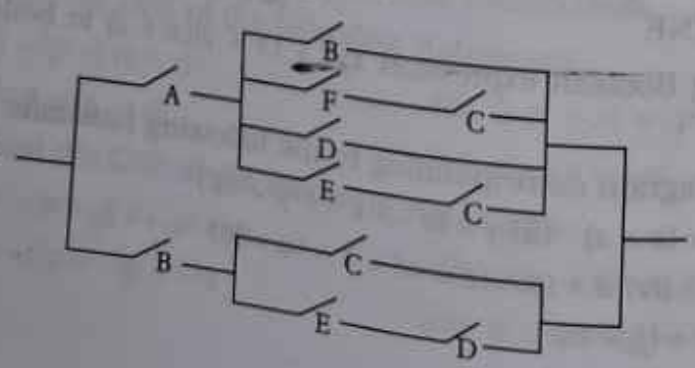


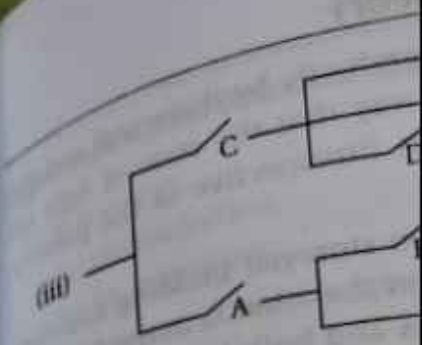
32. Write the Boolean expression for each of these following relay logic circ showing how the output algebraically relates to the inputs :

(i)



(ii)





33. A committee of three per
B has voting weight 1 ar
press a button to cast hi
glow when a majority o

34. Write the equivalent stat
the simplest equivalent

35. A committee of three
member can press a
that light will glow wh

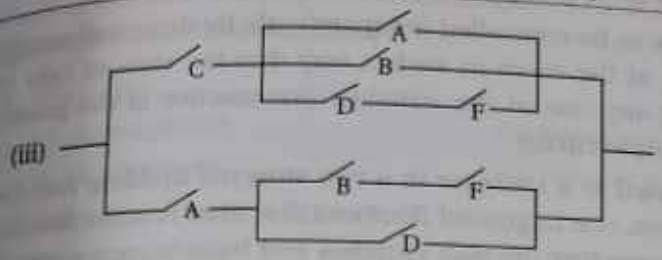
36. A committee of six pe
member can press a b
the light will glow whe

37. We want that a light
wall switches located
of any one of the sw
OFF to ON). Design

38. A committee of three
member can press t
in which current wil

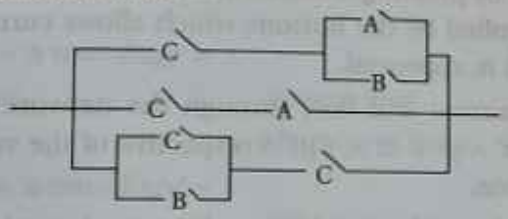
39. Design a switching c
such a way that eithe
of the state of the o

40. Design a simple circ
such a way that eithe
(irrespective of the



33. A committee of three persons A, B, C decides proposals by majority of votes. B has voting weight 1 and each of A and B has voting weight 2. Each can press a button to cast his vote. Design a simple circuit so that the light will glow when a majority of votes is cast in favour of the proposal.

34. Write the equivalent statement form representing the following circuit. Find the simplest equivalent circuit. [CU, 2021]



35. A committee of three persons decides proposal by majority of votes. Each member can press a bottom to cast his vote. Design a simple circuit so that light will glow when majority of vote is cast in favour of a proposal. [CH, 2011]

36. A committee of six persons decides proposals by a majority of votes. Each member can press a button to cast his vote. Design a simple circuit so that the light will glow when a majority of votes is cast in favour of the proposal.

37. We want that a light in a room will be controlled independently by three wall switches located at three doors of a room in such a way that flicking of any one of the switch will change the state of light (ON to OFF and OFF to ON). Design a switching circuit which will do the required job. [CH, 2005, 2012]

38. A committee of three persons decides proposals by a majority of votes. Each member can press to signify 'yes' vote. Design a simple switching circuit in which current will flow when and only when a majority will be 'yes'.

39. Design a switching circuit connecting to wall switches and a light bulb in such a way that either switch may be used to control the light independently of the state of the other.

40. Design a simple circuit connecting to wall switches and two light bulbs in such a way that either switch can be used to control the light independently (irrespective of the state of others).

41. A light in a room is to be controlled independently by three wall switches at three entrances of the room in such a way that the state of light change by flicking any one of the switches (irrespective of the pre... state). Design a simple circuit.

42. A light bulb is located at a staircase in a two storeyed building and there are two wall switches, one in ground floor and the other is in the first ... Design a circuit connecting the wall switches and bulb in such a way ... either switch may be used to control the light independently of the ... of other.

43. A committee consists of the president, vice-president, secretary ... treasurer. A proposal is approved if and only if it receives a majority ... or the vote of the president plus one other member. Each member appro... the proposal by pressing a button attached to their seats. Design a switch... circuit controlled by the buttons which allows current to pass if and ... if a proposal is approved.

44. Show that current will flow through the network given by the funct... $[(x \wedge y) \wedge ((x' \wedge y) \vee (x \wedge y'))]'$ irrespective of the switches x and / or ... closed or open.

45. Draw a switching circuit which realizes the logical expressions
(i) $((x \wedge y) \vee (x \wedge z) \vee (x' \wedge z') \wedge z' \wedge (x \vee y \vee z))$
(ii) $(x \vee y) \wedge [z \vee x' \wedge (y' \vee z')]$
(iii) $(x \wedge y \wedge z') \vee (x \wedge y' \wedge z) \vee (x' \wedge y' \wedge z')$

46. Draw the switching circuit which realizes the Boolean expression
(i) $[((x \wedge y) \vee (x \wedge z) \vee (x' \wedge z') \vee (z \wedge x))] \vee (y \wedge y')$
(ii) $(a \vee b) \wedge (a' \vee (c \wedge b')] \vee (b \vee c') \wedge a$
(iii) $(x \wedge y) \vee y' \wedge [(x' \wedge x) \wedge (y \wedge z')]$
Find its equivalent simplest circuit.

47. Show that $(P \rightarrow (Q \vee R))$ is a logical consequence of $(P \rightarrow Q)$. [CU, 20...

48. State Deduction Theorem in propositional logic.
Show that $\{B \rightarrow C, C \rightarrow D\} \rightarrow B \rightarrow D$ where B, C, D are well formed formu... in propositional logic. [CU, 20...

49. Show that the well formed formula $B \rightarrow (C \rightarrow (B \wedge C))$ is a theorem ... propositional logic, where B, C are well defined formulas in propositi... logic. [CU, 20...

50. Determine whether the following propositional formula is satisfiable ...
$(P \rightarrow (Q \rightarrow R)) \wedge (Q \rightarrow \sim R)$ [CU, 20...

51. Convert DNF into CNF
$f(p, q, r) = (\sim p \wedge q \wedge r) \vee (p \wedge q \wedge \sim r) \vee (\sim p \wedge q \wedge \sim r) \vee (p \wedge \sim q \wedge r)$ [MU, 2...

52. Write down the negation of the following proposition "for every numb... there is a number y such that $y < x$" [MU, 2...

Multiple Choice Questions (MCQ)

1. If $p \vee \sim p \equiv$
(a) contradiction
(c) an argument

2. The proposition $p \wedge (q \vee \sim q)$ is
(a) contradiction
(c) both (a) and (b)

3. A statement T is called tautol...
(a) T is true for all possible val...
(b) T is false for all possible va...
(c) T is true as well as false fo...
(d) none of these

4. 'any proposition' ∨ 'a tautol...
(a) the proposition
(c) contradiction

5. 'any proposition' ∧ 'a tautol...
(a) the proposition
(c) contradiction

6. In the argument $p \wedge q, \sim p$ ...
(a) principal proposition
(c) conclusion

7. In the argument $\sim p, q \wedge r,$
(a) 1 (b) 2

8. In the argument $\sim p, \sim p \vee$
(a) $\sim q$ (b) p

9. Which of the following is
(a) $(a \vee b) \rightarrow (b \wedge c)$
(c) $(a \vee b) \rightarrow (b \rightarrow c)$

10. In the proposition $p \vee (\sim$
(a) a tautology
(c) p

11. Let p, q, r and s be fo... arguments :
P : $\{(\sim p \vee q) \wedge (r \rightarrow s) \wedge$
Q : $\{(\sim p \vee q) \vee (q \rightarrow (p$
R : $[\{(q \wedge r) \rightarrow p\} \wedge (\sim q$
S : $[p \wedge (p \rightarrow r) \wedge (q \wedge \sim$

Multiple Choice Questions (MCQ)

1. If $p \vee \sim p \equiv$
   (a) contradiction (b) Tautology
   (c) an argument (d) none of these

2. The proposition $p \wedge (q \vee \sim q)$ is [MAKUT 2006]
   (a) contradiction (b) Tautology
   (c) both (a) and (b) (d) none of these

3. A statement T is called tautology if [MAKUT 2006]
   (a) T is true for all possible values of its variable
   (b) T is false for all possible values of its variable
   (c) T is true as well as false for few possible values of its variables
   (d) none of these

4. 'any proposition' $\vee$ 'a tautology' $\equiv$
   (a) the proposition (b) tautology
   (c) contradiction (d) none of these

5. 'any proposition' $\wedge$ 'a tautology' $\equiv$
   (a) the proposition (b) tautology
   (c) contradiction (d) none of these

6. In the argument $p \wedge q, \sim p \to q, p \wedge q$ is a
   (a) principal proposition (b) premises
   (c) conclusion (d) none of these

7. In the argument $\sim p, q \wedge r, r \to p, p \leftrightarrow q \to q \to r$ the number of premises is
   (a) 1 (b) 2 (c) 3 (d) 4

8. In the argument $\sim p, \sim p \vee q, \sim q \to p \leftrightarrow q$ the conclusion is
   (a) $\sim q$ (b) p (c) q (d) $p \leftrightarrow q$

9. Which of the following is/are tautology?
   (a) $(a \vee b) \to (b \wedge c)$ (b) $(a \wedge b) \to (b \vee c)$
   (c) $(a \vee b) \to (b \to c)$ (d) $(a \to b) \to (b \to c)$

10. In the proposition $p \vee (\sim p \vee q)$ is
    (a) a tautology (b) a contradiction
    (c) p (d) q

11. Let p, q, r and s be four primitive statements. Consider the following arguments :

    P : $[(\sim p \vee q) \wedge (r \to s) \wedge (p \vee r)] \to (\sim s \to q)$

    Q : $[(\sim p \vee q) \vee (q \to (p \to r))] \to \sim r$

    R : $[\{(q \wedge r) \to p\} \wedge (\sim q \vee p)] \to r$

    S : $[p \wedge (p \to r) \wedge (q \wedge \sim r)] \to q$

Which of the above arguments are valid?
(a) P and Q only (b) P and R only
(c) P and S only (d) P, Q and S

12. Consider the following propositional statements :
$P_1 : [(A \wedge B) \rightarrow C] \equiv [(A \rightarrow C) \wedge (B \rightarrow C)]$
$P_2 : [(A \vee B) \rightarrow C] \equiv [(A \rightarrow C) \vee (B \rightarrow C)]$
Which of the following is TRUE?
(a) P_1 is a tautology but not P_2 (b) P_2 is a tautology but not
(c) P_1 and P_1 are both tautologies (d) both P_1 and P_1 are not

13. Consider the following logical inferences.
P_1 : If there is rain then the football match will not be played.
The football match was played.
Inference : There was no rain.
P_2 : If it rains then the cricket match will not be played.
It did not rain.
Inference: The football match was played.
Which of the following is TRUE?
(a) both P_1 and P_2 are correct inferences.
(b) P_1 is correct but P_2 is not correct.
(c) P_1 is not correct but P_2 is correct inference.
(d) both P_1 and P_2 are not correct inferences.

14. Consider the following statements :
P : Good mobile phones are not cheap
Q : Cheap mobile phones are not good
L : P implies Q
M : Q implies P
N : P is equivalent to Q
which one of the following about L, M and N is correct?
(a) only L is TRUE (b) only M is TRUE
(c) only N is TRUE (d) L,M and N are TRUE

15. Let p, q and r be propositions and the expression $(p \rightarrow q)$ contradiction. Then the expression $(r \rightarrow p) \rightarrow q$ is
(a) a tautology (b) a contradiction
(c) always TRUE when p is FALSE (d) always TRUE when q is

16. $p \rightarrow (p \vee q)$ is a
(a) tautology (b) contradiction
(c) contingency (d) none of these

17. $p \rightarrow (q \wedge \neg q) \equiv$
(a) $\neg p$ (b) $\neg q$

18. From the hypothesis $p \rightarrow (p \wedge$
(a) $p \rightarrow q$ (b) $q \rightarrow p$

19. $(p \rightarrow q) \leftrightarrow p$ is a
(a) tautology (b) contrad

20. From the premises $\neg p$ and p
(a) q (b) p

21. From the premises p and p
(a) p (b) q

22. From the premises $\neg q$ and
(a) p (b) q

23. $(p \rightarrow q) \leftrightarrow (\neg p \vee q)$ is a
(a) tautology
(c) contingency

24. P, Q and R are three at
$(P \vee Q) \rightarrow R$ and Y denote
Which one of the followin
(a) $X \vee Y$ (b) $X \rightarrow$

25. The number of different
statement letters is
(a) 2^{n^2} (b) 2^n

26. An one element adequat
(a) $\{\wedge\}$
(c) $\{/\}$

27. Minimal functionally co
(a) $\{\neg, \vee, \wedge\}$
(c) $\{\neg, \wedge\}$

28. Consider the statement
Jane is Bill's niece" and
statements is equivaler
(a) If Bill is Tom's broth
niece
(b) If Bill is not Tom's br
niece.
(c) If Bill is not Tom's br
(d) If Bill is Tom's broth

17. $p \to (q \wedge \neg q) \equiv$
(a) $\neg p$ (b) $\neg q$ (c) F (d) T

18. From the hypothesis $p \to (p \wedge q)$, one can conclude
(a) $p \to q$ (b) $q \to p$ (c) p (d) q

19. $(p \to q) \leftrightarrow p$ is a
(a) tautology (b) contradiction (c) contingency (d) none of these

20. From the premises $\neg p$ and $p \vee q$, one can conclude
(a) q (b) p (c) $\neg p$ (d) $\neg q$

21. From the premises p and $p \to q$, one can conclude
(a) p (b) q (c) $\neg p$ (d) $\neg q$

22. From the premises $\neg q$ and $p \to q$, one can conclude
(a) p (b) q (c) $\neg p$ (d) $\neg q$

23. $(p \to q) \leftrightarrow (\neg p \vee q)$ is a
(a) tautology (b) contradiction
(c) contingency (d) none of these

24. P, Q and R are three atomic propositional assertions, Let X denote $(P \vee Q) \to R$ and Y denotes $(P \to R) \vee (Q \to R)$.
Which one of the following is a tautology?
(a) $X \vee Y$ (b) $X \to Y$ (c) $Y \to X$ (d) $\neg Y \to X$

25. The number of different non-equivalent statement formula containing statement letters is [CU, 2021]
(a) 2^{n^2} (b) 2^n (c) 2^{2^n} (d) 2^{2n}

26. An one element adequate system of connective is
(a) $\{\wedge\}$ (b) $\{\vee\}$
(c) $\{/\}$ (d) $\{\neg\}$

27. Minimal functionally complete set of connectives is
(a) $\{\sim, \vee, \wedge\}$ (b) $\{\sim, \vee\}$
(c) $\{\sim, \wedge\}$ (d) both (b) and (c)

28. Consider the statement form $p \to q$ where p = "If Tom is Jane's father then Jane is Bill's niece" and q = "Bill is Tom's brother". Which of the following statements is equivalent to this statement
(a) If Bill is Tom's brother, then Tom is Jane's father and Jane is not Bill's niece
(b) If Bill is not Tom's brother, then Tom is Jane's father and Jane is not Bill's niece.
(c) If Bill is not Tom's brother, then Tom is Jane's father or Jane is Bill's niece.
(d) If Bill is Tom's brother, then Tom is Jane's father and Jane is Bill's niece.

29. Consider the statement, "If n is divisible by 30, then n is divisible by 3 and by 5". Which of the following statements is equivalent to this statement

(a) If n is not divisible by 30 then n is divisible by 2 or divisible by 3 or divisible by 5.

(b) If n is not divisible by 30 then n is not divisible by 2 or not divisible by 3 or not divisible by 5

(c) If n is divisible by 2 and divisible by 3 and divisible by 5 then n is divisible by 30.

(d) If n is not divisible by 2 or not divisible by 3 or not divisible by 5 then n is not divisible by 30.

30. Which of the following statements is the contrapositive of the statement "You win the game if you know the rules but you are not overconfident."

(a) If you lose the game then you don't know the rules or you are overconfident.

(b) A sufficient condition that you win the game is that you know the rules or you are not overconfident.

(c) If you don't know the rules or are overconfidnet you lose the game

(d) If you know the rules and are overconfident then you win the game

31. The statement form $(p \leftrightarrow r) \rightarrow (q \leftrightarrow r)$ is equvalent to

(a) $[(\sim p \vee r) \wedge (p \vee \sim r)] \vee [(\sim q \vee r) \wedge (q \vee \sim r)]$

(b) $\sim[(\sim p \vee r) \wedge (p \vee \sim r)] \wedge [(\sim q \vee r) \wedge (q \vee \sim r)]$

(c) $[(\sim p \vee r) \wedge (p \vee \sim r)] \wedge [(\sim q \vee r) \wedge (q \vee \sim r)]$

(d) $\sim[(\sim p \vee r) \wedge (p \vee \sim r)] \vee [(\sim q \vee r) \wedge (q \vee \sim r)]$

32. Consider the statement, "Given that people who are in need of refuge and consolation are apt to do odd things. It is clear that people who are apt to do odd things are in need of refuge and consolation." This statement, of the form $(P \rightarrow Q) \rightarrow (Q \rightarrow P)$, is logically equivalent to

(a) People who are in need of refuge and consolation are not apt to do odd things

(b) People are apt to do odd things if and only if they are in need of refuge and consolation

(c) People who are apt to do odd things are in need of refuge and consolation

(d) People who are in need of refuge and consolation are apt to do odd things

(e) People who aren't apt to do odd things are not in need of refuge and consolation

33. Which of the following statements is the negation of "There exists either a computer or a person who knows both discrete maths and Java."

(a) There exists a person who is not discrete maths and Jave or there and who knows both discrete

(b) There exists a person who is who is a mathematician who

(c) There exists a person who discrete maths and Java both discrete maths and

(d) There exists a computer Java or there exists a per discrete maths and Java.

(e) There exists a person who who knows both discret

A. Short and Long Answer Type

1. (i) Tautology (ii) Tautolog (v) Contradiction (vi) Tau

5. (i) $p \vee (q \wedge r)$ (ii) $-p \vee -$ (iv) $((\sim p \wedge q) \vee (q \vee \sim r)) \wedge$

6. (i) The rule of Inference i (ii) The rule of Inference (iii) The rule of Inference (iv) The rule of Inference (v) The rule of Inference (vi) The rule of Inference

7. (i) not valid (ii) valid

8. (i) valid (ii) valid (iii) va

17. (i) $(p \wedge \sim q) \vee (p \wedge \sim r)$ (ii) $(p \wedge q) \vee (p \wedge q \wedge r)$ (iii) $(p \wedge \sim p) \vee (p \wedge q)$ (iv) $p \wedge q$

18. (i) $(p \vee \sim q) \wedge (\sim p \vee q)$ (iii) $(\sim p \vee r) \wedge (q \vee r)$

33. **Which of the following statements is NOT equivalent to the statement, "There exists either a computer scientist or a mathematician who knows both discrete maths and Java."**

(a) There exists a person who is a computer scientist and who knows both discrete maths and Jave or there exists a person who is a mathematician and who knows both discrete maths and Java.

(b) There exists a person who is a computer scientist or there exists a person who is a mathematician who knows discrete maths or who knows Java.

(c) There exists a person who is a computer scientist and who knows both discrete maths and Java or there exists a mathematician who knows both discrete maths and Java.

(d) There exists a computer scientist who knows both discrete maths and Java or there exists a person who is a mathematician who knows both discrete maths and Java.

(e) There exists a person who is a computer scientist or a mathematician who knows both discrete maths and Java.

Answers

A. Short and Long Answer Type Questions

1. (i) Tautology (ii) Tautology (iii) Contradiction (iv) Contradiction (v) Contradiction (vi) Tautology (vii) Contradiction

5. (i) $p \vee (q \wedge r)$ (ii) $\neg p \vee \neg q$ (iii) $(p \vee \neg q) \vee (\neg p \vee q)$ (iv) $((\neg p \wedge q) \vee (q \vee \neg r)) \wedge (p \wedge \mathrm{T})$, where T is true

6. (i) The rule of Inference is "Simplification"
(ii) The rule of Inference is "Simplification"
(iii) The rule of Inference is "Modus ponens"
(iv) The rule of Inference is "Addition"
(v) The rule of Inference is "Hypothetical Syllogism"
(vi) The rule of Inference is "Disjunctive Syllogism"

7. (i) not valid (ii) valid (iii) valid (iv) valid (v) not valid (vi) not valid

8. (i) valid (ii) valid (iii) valid (iv) valid (v) valid (vi) valid

17. (i) $(p \wedge \neg q) \vee (p \wedge \neg r)$
(ii) $(p \wedge q) \vee (p \wedge q \wedge r) \vee (\neg p \wedge \neg q \wedge r)$
(iii) $(p \wedge \neg p) \vee (p \wedge q)$
(iv) $p \wedge q$

18. (i) $(p \vee \neg q) \wedge (\neg p \vee q) \wedge (p \vee q)$ (ii) $(\neg p \vee \neg r) \wedge (q \vee r)$
(iii) $(\neg p \vee r) \wedge (q \vee r)$ (iv) $(\neg p \vee \neg q) \wedge (p \wedge q)$

19. (i) $p \wedge q$ (ii) $(p \wedge \sim q) \vee (\sim p \wedge q)$ (iii) $p \wedge q$

(iv) The PDNF of the given statement cannot be found as the given ... is a contradiction

20. (i) $(\sim p \vee q) \wedge (p \vee \sim q)$ (ii) $(\sim p \vee q) \wedge (p \vee \sim q)$ (iii) $(\sim p \vee q) \wedge (p \vee \sim$

(iv) $(\sim p \vee q) \wedge (\sim p \vee q) \wedge (p \vee \sim q) \wedge (p \vee q)$

(v) The PCNF of the given statement cannot be found as the given ... is a Tautology

21. (i) PDNF: $(p \wedge q) \vee (p \wedge \sim q) \vee (\sim p \wedge q) \vee (\sim p \wedge \sim q)$

PCNF of the statement is not possible to determine

(ii) PDNF of the statement is not possible to determine

PCNF: $(\sim p \vee \sim q) \wedge (\sim p \vee q) \wedge (p \vee q)$

(iii) PDNF: $(p \wedge q \wedge r) \vee (p \wedge q \wedge \sim r) \vee (p \wedge \sim q \wedge r) \vee (\sim p \wedge q \wedge r)$

PCNF: $(\sim p \vee q \vee r) \wedge (p \vee \sim q \vee r) \wedge (p \vee q \vee \sim r) \wedge (p \vee q \vee r)$

(iv) PDNF: $(p \wedge q \wedge r) \vee (p \wedge q \wedge \sim r) \vee (\sim p \wedge q \wedge r) \vee (\sim p \wedge q \wedge \sim r)$

PCNF: $(\sim p \vee q \vee \sim r) \wedge (\sim p \vee q \vee r) \wedge (p \vee q \vee \sim r) \wedge (p \vee q \vee r)$

22. (i) PDNF: $(p \wedge q) \vee (p \wedge \sim q) \vee (\sim p \wedge q) \vee (\sim p \wedge \sim q)$

PCNF of the statement is not possible to determine

(ii) PDNF of the statement is not possible to determine

PCNF: $(\sim p \vee \sim q) \wedge (\sim p \vee q) \wedge (p \vee q)$

(iii) PDNF: $(p \wedge q \wedge r) \vee (p \wedge q \wedge \sim r) \vee (p \wedge \sim q \wedge r) \vee (\sim p \wedge q \wedge r)$

PCNF: $(\sim p \vee q \vee r) \wedge (p \vee \sim q \vee r) \wedge (p \vee q \vee \sim r) \wedge (p \vee q \vee r)$

(iv) PDNF: $(p \wedge q \wedge r) \vee (p \wedge q \wedge \sim r) \vee (\sim p \wedge q \wedge r) \vee (\sim p \wedge q \wedge \sim r)$

PCNF: $(\sim p \vee q \vee \sim r) \wedge (\sim p \vee q \vee r) \wedge (p \vee q \vee \sim r) \wedge (p \vee q \vee r)$

. Yes

25. No

28. CNF : $(x' + y' + z')(x + y + z)(x + y' + z')(x' + y' + z)(x + y + z')$

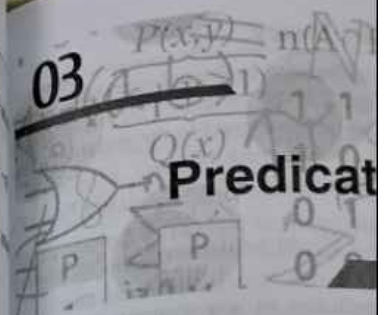
DNF: $xy'z + xy'z' + x'yz'$

29. CNF : $(x + y + z)(x + y + z')(x + y' + z)$

DNF: $xyz + x'yz + xy'z + xyz' + xy'z'$

B. Multiple Choice Questions (MCQ)

1. b	2. a	3. a	4. b	5. a	6. b	7. d	8. d	9.
11. c	12. d	13. b	14. d	15. d	16. a	17. a	18. d	19.
21. c	22. c	23. a	24. b	25. b	26. d	27. d	28. b	29.
31. d	32. c	33. c						



03 Predicat

Introduction

In the previous chapter we hac propositional logic is not powerful e can not deal with certain types of re propositions. For example, the asse proposition as the statement is neit the value of x. So, such sentences ca In this case we need a stronger logic

The predicate calculus is an ex includes the notion of quantificatio which have a definite truth-value, predicates, which are assertions in presented as "ranging" over some that applies to these variables. State in which there are either no variabl quantified. All of the operations of th virtually without change. What nee operations interact with quantificati

3.1 . Predicate Calculus

In mathematics and computer p variables such as "$x > 6$", "$x = y + z$" true nor false, when the values of th

The statement "x is greater than 6 is the subject of the statement. The s the property which a subject can hav the word (or set of words) which de the predicate.

03

Predicate Calculus

Introduction

In the previous chapter we had a total view of propositional logic. The propositional logic is not powerful enough to handle all types of assertions and can not deal with certain types of relationships such as equivalence relation in propositions. For example, the assertion '$x < 6$', where x is a variable, is not a proposition as the statement is neither true nor false unless and until we know the value of x. So, such sentences can not be handled by propositional calculus. In this case we need a stronger logic which can deal with such assertions.

The predicate calculus is an extension of the propositional calculus that includes the notion of quantification. Instead of dealing only with statements, which have a definite truth-value, we deal with the more general notion of predicates, which are assertions in which variables appear. These variables are presented as "ranging" over some given sets, and quantification is a process that applies to these variables. Statements are here viewed as special predicates in which there are either no variables at all or in which all variables have been quantified. All of the operations of the propositional calculus extend to predicates virtually without change. What needs to be understood, however, is how these operations interact with quantification.

3.1 Predicate Calculus or Predicate Logic

In mathematics and computer programs, we encounter statements involving variables such as "$x > 6$", "$x = y + z$" and "$x + y = z$". These statements are neither true nor false, when the values of the variables are not specified.

The statement "x is greater than 6" has two parts. The first part, the variable x, is the subject of the statement. The second part "is greater than 6", which refers to the property which a subject can have, is called the *predicate*. Thus, in a sentence, the word (or set of words) which describe the nature or properties of a subject is the predicate.

The statement "x is greater than 6" can be denoted by the notation P(x), where P denotes the predicate "is greater than" and x is the variable. P(x) is called a propositional function at x. Once a value has been assigned to the variable x, the statement P(x) becomes a proposition and assumes truth values.

For example, the truth value of P (10) ($\equiv 10 > 6$) and P(3) ($\equiv 3 > 6$) are True (T) and False (F) respectively.

The statements "$x = y + 2$" and "$x + y = z$" will be denoted by P(x, y) and P(x, y, z) respectively.

The logic based on the analysis of predicates in any statement is called *predicate logic* or *predicate calculus.*

3.2 Domain (Universe of discourse) of a proposition function

The set of all possible values that may be substituted in place of the variable in a proposition function is called *domain (Universe of discourse)* of the proposition function.

Example 3.1 : For P(x): "$x > 6$" we assume that x takes values from A = {1, 3, 10, 12}.

Here, A is the universe of discourse. Here, P(1) and P(3) are false (F) while P(10) and P(12) are true (T).

Example 3.2 : Let us consider that the domain or Universe of discourse is "set of all human beings". Now, let us consider the proposition "Nilanjan is a doctor". Then, the propositional function P(x) is "x is a doctor". Here, P is the proposition "Nilanjan" and the predicate is "is a doctor".

Example 3.3 : Let us consider the proposition function P(x, y) which is "$x^2 + y^2 = 5$". Let the Universe of discourse P(x, y) be R × R, where R is the set of all real numbers. The proposition function is defined for two variables x and y. Here, P(1, 2) is true (T) but P(1, 3) is false (F).

3.3 Quantifiers

We have seen that a proposition can be created from a propositional function P(x) by assigning particular value to the variable x. *Quantification* is the process of creating a proposition from a proposition function. There are two types of quantifications, namely, *Universal quantification* and *Existential quantification*.

3.3.1 Universal Quantification

The universal quantification of the proposition function P(x) is the proposition "P(x) is true for all values of x in the domain". This quantification is denoted by $\forall x$, P(x). The symbol $\forall$ is called the Universal quantifier.

Example 3.4 : Let P(x) be the propositional function "$x \in$ {1, 3, 5, 7, 9, 11, 13, 15}. Then, the proposition P(14) is not true for $x = 14$" $\forall x \in$ {1, 3, 5, 7, 9, 11, 13, 15}. The proposition P(x) is true for all values of x in the domain A = {1, 3, 5, 7, 9, 11, 13, 15}. But, if the domain is taken A = {1, 3, 5, 7, 9, 11, 13, 14 do not satisfy the relation P(x) is not true for all values of x". Obviously the proposition P(x) is true for $x = 2$.

Example 3.5 : Let P(x) be the proposition "$x > 3$" and the domain be A: "set of all real numbers". Then, the Universal quantification $\forall x$, P(x) is the proposition "All human beings are mortal". Obviously the proposition P(x) is not true for $x = 2$.

Example 3.6 : Let P(x) be the proposition "$x > 3$" and the domain be A: "set of all human beings". Obviously the proposition P(x) is true for all values of x". Obviously the proposition P(x) is not true for $x = 2$.

Example 3.7 : Let P(x) be the proposition "$x > 3$" and the domain be A: "set of all real numbers". Then, the Universal quantification $\forall x$, P(x) is the statement "All human beings are mortal". Obviously the proposition P(x) is not true for $x = 2$.

Example 3.8 : Let P(x) be the proposition "$x > 3$" and the domain be A: "set of all real numbers". Then, the Universal quantification $\forall x$, P(x) is the statement "All human beings are mortal". Obviously the proposition P(x) is not true for $x = 2$.

3.3.2 Existential quantification

The Existential quantification of the proposition "There exists a value of x This quantification is denoted by $\exists x$, P(quantifier.

Example 3.8 : Let P(x) be the proposition "$x > 3$" and the domain be A = {1, 3, 5, 7, 9, 11, 13, 15}. Then, the Existential quantification of the proposition "There exists a value of x for which $x > 3$" is true. This quantification is denoted by $\exists x$, P(x). $\exists x$ P(x) is true (T).

But, if the domain is taken as A = { since there is no value of x for which

Example 3.9 : Let P(x) be the proposition "$x > 3$" and the domain be A: "set of all real numbers". Then, the Existential quantification of the proposition "There exists a value of x for which $x > 3$" is true. This quantification is denoted by $\exists x$, P(x). $\exists x$ P(x) is true (T), since $x > 3$ is

Example 3.10 : Let P(x) be the proposition "$x > 3$" and the domain be A: "set of all real numbers". Then, the Existential quantification of the proposition "There exists a value of x for which $x > 3$" is true. This quantification is denoted by $\exists x$, P(x). $\exists x$ P(x) is true (T), since $x > 3$ is true for all values of x". Obviously the proposition P(x) is not true for $x = 2$. where the domain A is "set of all real numbers". Then, the Existential quantification $\exists x$, P(x) is the statement "Nilanjan is a good batsman".

Example 3.11 : Let P(x) be the proposition "$x > 3$" and the domain be A: "set of all real numbers". Then, the Existential quantification of the proposition "There exists a value of x for which $x > 3$" is true. This quantification is denoted by $\exists x$, P(x). $\exists x$ P(x) is false (F), since there $x + 5 = x$.

Example 3.4 : Let P(x) be the propositional function "$x + 3 < 14$". Let the domain be A = {1, 3, 5, 7, 9, 11, 13, 15}. Then, the Universal quantification is "$x + 3 < 14$" $\forall x \in$ {1, 3, 5, 7, 9, 11, 13, 15}. The proposition $\forall x$, P(x) is false (F), since the values 11, 13, 14 do not satisfy the relation "$x + 3 < 14$".

But, if the domain is taken A = {1, 3, 5, 6, 7, 9}, then $\forall x$, P(x) will be true.

Example 3.5 : Let P(x) be the propositional function "$x > 3$". Let the domain be A : "set of all real numbers". Then, the Universal quantification $\forall x$, P (x) is "$x > 3$ is true for all values of x". Obviously the proposition $\forall x$, P(x) is false (F), since $x > 3$ is not true for $x = 2$.

Example 3.6 : Let P(x) be the propositional function "x is mortal". Let the domain be A: "set of all human beings". Then the Universal quantification is "All human beings are mortal". Obviously the proposition $\forall x$, P(x) is true (T).

Example 3.7 : Let P(x) be the propositional function "x is a good batsman" where the domain A is "set of all students in a class". Then the Universal quantification $\forall x$, P(x) is the statement "Every student in the class is a good batsman".

3.3.2 Existential quantification

The Existential quantification of the propositional function P(x) is the proposition "There exists a value of x in the domain for which P(x) is true". This quantification is denoted by $\exists x$, P(x). The symbol $\exists$ is called the Existential quantifier.

Example 3.8 : Let P(x) be the propositional function "$x + 3 < 14$". Let the domain be A = {1, 3, 5, 7, 9, 11, 13, 15}. Then, the Existential quantification $\exists x$, P(x) is "there exists a value of x for which $x + 3 < 14$ is true". Obviously the proposition $\exists x$, P(x) is true (T).

But, if the domain is taken as A = {11, 13, 15}, then $\exists x$, P(x) will be false (F), since there is no value of x for which "$x + 3 < 14$" is true.

Example 3.9 : Let P(x) be the propositional function "$x > 3$". Let the domain be A: "set of all real numbers". Then, the Existential quantification $\exists x$, P(x) is "there exists a value of x for which $x > 3$ is true". Obviously the proposition $\exists x$, P(x) is true (T), since $x > 3$ is true for $x = 4$.

Example 3.10 : Let P(x) be the propositional function "x is a good batsman" where the domain A is "set of all students in a class". Then the existential quantification $\exists x$, P(x) is the statement "there exists a student in the class who is a good batsman".

Example 3.11 : Let P(x) be the propositional function "$x + 5 = x$". Let the domain be A: "set of all real numbers". Then, the Existential quantification $\exists x$, P(x) is "there exists a value of x for which $x + 5 = x$ is true". Obviously the proposition $\exists x$, P(x) is false (F), since there is no real value which satisfies the relation $x + 5 = x$.

3.4 Negation of a Quantified Expression

If $P(x)$ is the proposition statement "x has studied mathematical logic" $\forall x, P(x)$ means that "every student (in the class) has studied mathematical logic". The negation of this statement is "It is not the case that every student in the class has studied mathematical logic" or equivalently "there is a student in the class who has not studied mathematical logic" which is denoted by $\forall x, \neg P(x)$. Thus we can write

$$\neg(\forall x\, P(x)) \equiv \exists x\, (\neg P(x)),$$ for $x \in A$, A is the Universe of discourse

Similarly, $\exists x, P(x)$ means that "there is a student in the class who has studied mathematical logic". The negation of this statement is "Every student in the class has not studied mathematical logic" which is denoted by $\forall x, \neg P(x)$. Thus we write

$$\neg(\exists x\, P(x)) \equiv \forall x(\neg P(x)),$$ for $x \in A$, A is the Universe of discourse

The rules for negations for quantifiers are called De Morgan's laws for quantifiers.

The following table shows quantified statements and their negations :

Quantifications	Negation of Quantifications
$(\forall x)P(x)$ (True for all x)	$(\exists x)\ \neg P(x)$ (At least one x is false)
$(\forall x)\ \neg P(x)$ (False for all x)	$(\exists x)\ P(x)$ (At least one x is true)
$(\exists x)\ P(x)$ (At least one x is true)	$(\forall x)\ \neg P(x)$ (False for all x)
$(\exists x)\ \neg P(x)$ (At least one x is false)	$(\forall x)\ P(x)$ (True for all x)

Example 3.12 : Negate the following propositions :

(i) All men can run faster than all women.

(ii) Some women are more intelligent than all men.

(iii) Some animals do not live in forest.

(iv) All students pass in the semester examination.

Solution : (i) Some men can run faster than some women.

(ii) All women are more intelligent than some men.

(iii) All animals live in forest.

(iv) Some students do not pass in the semester examination.

Example 3.13 : Negate the following propositions :

(i) $\forall x\, P(x) \wedge \exists y\, Q(y)$ **(ii)** $\forall x\, P(x) \vee \exists y\, Q(y)$ **(iii)** $\exists x\, P(x) \wedge \forall y\, Q(y)$

(iv) $\exists x\, P(x) \vee \forall y\, Q(y)$ **(v)** $(\exists x \in A)\ (x + 6 = 25)$ **(vi)** $(\forall x \in A)\ (x < 25)$

Solution : (i) $\neg(\forall x\, P(x) \wedge \exists y\, Q(y))$

$\equiv \neg \forall x\, P(x) \vee \neg \exists y\, Q(y)$, by De Morg

$\equiv \exists x \neg P(x) \vee \forall y\ \neg Q(y)$

(ii) $\neg(\forall x\, P(x) \vee \exists y\, Q(y))$

$\equiv \neg \forall x\, P(x) \wedge \neg \exists y\, Q(y)$, by De Mo

$\equiv \exists x \neg P(x) \wedge \forall y \neg Q(y)$

(iii) $\neg(\exists x\, P(x) \wedge \forall y\, Q(y))$

$\equiv \neg \exists x\, P(x) \wedge \neg \forall y\, Q(y)$, by De Mo

$\equiv \forall x \neg P(x) \wedge \exists y \neg Q(y)$

(iv) $\neg(\exists x\, P(x) \vee \forall y\, Q(y))$

$\equiv \neg \exists x\, P(x) \wedge \neg \forall y\, Q(y)$, by De M

$\equiv \forall x \neg P(x) \wedge \exists y \neg Q(y)$.

(v) $\neg(\exists x \in A)(x + 6 = 25)$

$\equiv \forall x \in A \neg((x + 6) = 25)$

$\equiv (\forall x \in A)((x + 6) \neq 25)$

(vi) $\neg(\forall x \in A)\ (x < 25)$

$\equiv \exists x \in A \neg(x < 25)$

$\equiv (\exists x \in A)\ (x \geq 25)$

3.5 Nested (more th

There are situations when qu
1-place or n-place predicate for
1 or n variables). For example let

Now, $\forall x\, \forall y\, P(x,y) \equiv \forall x\, [\forall y\, P($

$\equiv \forall y\, [\forall x\, P($

And $\exists x \exists y\, P(x, y) \equiv \exists x\, [\exists y\, P($

From the meaning of quanti
hold good :

$\forall x\, \forall y\, P(x, y) \Rightarrow (\exists y)\ \forall x\, P($

$\forall y\, \forall x\, P(x, y) \Rightarrow (\exists y)\ \forall x\, P($

Note

The negation of multiply qua
applying the rules for negatio
Thus $\neg[\forall x \exists y P(x, y)\ ] \equiv \exists x\, [\neg \exists y$

Solution : (i) $\neg(\forall x\, P(x) \wedge \exists y\, Q(y))$

$\equiv \neg \forall x\, P(x) \vee \neg \exists y\, Q(y)$, by De Morgan's law

$\equiv \exists x \neg P(x) \vee \forall y\, \neg Q(y)$

(ii) $\neg(\forall x\, P(x) \vee \exists y\, Q(y))$

$\equiv \neg \forall x\, P(x) \wedge \neg \exists y\, Q(y)$, by De Morgan's law

$\equiv \exists x \neg P(x) \wedge \forall y \neg Q(y)$

(iii) $\neg(\exists x\, P(x) \wedge \forall y\, Q(y))$

$\equiv \neg \exists x\, P(x) \wedge \neg \forall y\, Q(y)$, by De Morgan's law

$\equiv \forall x \neg P(x) \wedge \exists y \neg Q(y)$

(iv) $\neg(\exists x\, P(x) \vee \forall y\, Q(y))$

$\equiv \neg \exists x\, P(x) \wedge \neg \forall y\, Q(y)$, by De Morgan's law

$\equiv \forall x \neg P(x) \wedge \exists y \neg Q(y)$,

(v) $\neg(\exists x \in A)(x + 6 = 25)$

$\equiv \forall x \in A \neg((x + 6) = 25)$

$\equiv (\forall x \in A)((x + 6) \neq 25)$

(vi) $\neg(\forall x \in A)\ (x < 25)$

$\equiv \exists x \in A \neg(x < 25)$

$\equiv (\exists x \in A)\ (x \geq 25)$

3.5 Nested (more than one) Quantifiers

There are situations when quantifiers occur in combinations in respect of 1-place or n-place predicate formulas (i.e., propositional functions containing 1 or n variables). For example let us consider a 2-place predicate formula $P(x,y)$.

Now, $\forall x\, \forall y\, P(x,y) \equiv \forall x\, [\forall y\, P(x, y)]$

$\equiv \forall y\, [\forall x\, P(x, y)]$ ... (1)

And $\exists x\, \exists y\, P(x, y) \equiv \exists x\, [\exists y\, P(x, y)] \equiv \exists y\, [\exists x\, P(x, y)$ ... (2)

From the meaning of quantifiers by (1) and (2) the following simplifications hold good :

$\forall x\, \forall y\, P(x, y) \Rightarrow (\exists y)\ \forall x\, P(x, y) \Rightarrow \forall x\, \exists y\, P(x, y)$

$\forall y\, \forall x\, P(x, y) \Rightarrow (\exists y)\ \forall x\, P(x, y) \Rightarrow \forall x\, \exists y\, P(x, y)$

Note

The negation of multiply quantified predicate formulas may be obtained by applying the rules for negation (given earlier) from left to right.

Thus $\neg[\forall x\, \exists y P(x, y)\ ] \equiv \exists x\, [\neg \exists y P(x, y)] \equiv \exists x \forall y [\neg P(x, y)]$

3.6 Parentheses

Sometimes we have very complicated expression, then parentheses w reading the expression more easier. The similar conventions about the omission of paranthesis in propositional calculus are also used in predic To make use of parentheses firstly we use the connectives ¬, ∧ and ∨ and need to take the quantifiers ∀ and ∃ and finally we deal with → and ↔ right. In the connectives such as negations and quantifications, we will from right to left.

Let us consider the following example to show how the Parenth restored via the following steps.

Example 3.14 : Express the following sentences using predicate, quan and parentheses.

"All students of Gogardanda Hindu college play basketball."

"Some students play hockey."

Solution : Let us assume

$P(x)$: x is a student.

$Q(x)$: x plays basketball.

$R(x)$: x plays hockey.

Then "Some students play hockey" can be expressed logi $\exists x(P(x) \wedge \neg R(x))$

Example 3.15 : Let us consider a formula $\forall x \exists y P(x, y)$. We can parentheses in the following steps :

Step 1 : $\forall x\, \exists y\, P(x, y)$

Step 2 : $(\forall x)(\exists y)\, P(x, y)$

Step 3 : $(\forall x)((\exists y)\, P(x, y))$

Step 4 : $((\forall x)((\exists y)\, P(x, y)))$

Example 3.16 : Let us consider a formula $\forall x\, P(x) \vee Q(x, y)$. We can parentheses in the following steps :

Step 1 : $\forall x\, P(x) \vee Q(x, y)$

Step 2 : $\forall x\, (P(x) \vee Q(x, y))$

Step 3 : $(\forall x)(P(x) \vee Q(x, y))$

Step 4 : $((\forall x)(P(x) \vee Q(x, y)))$

Example 3.17 : Let us consider a formula $\forall x \neg \exists y P(x, y)$. One can parentheses in the following steps :

Step 1 : $\forall x \neg \exists y\, P(x, y)$

Step 2 : $(\forall x)\, \neg (\exists y)\, P(x, y)$

Step 3 : $(\forall x)\, \neg ((\exists y)\, P(x, y))$

Step 4 : $(\forall x)(\neg ((\exists y)\, P(x, y)))$

Step 5 : $((\forall x)(\neg ((\exists y)\, P(x, y))))$

3.7 First order Logic

First-order logic can be understood as an extensio In propositional logic the atomic formulas have no int propositional variables that are either true or false. In fi formulas are predicates that assert a relationship an significant concept in first-order logic is quantificatio a certain property holds for all elements or that it hold First-order logic is also known as predicate logic or First-order logic is a powerful language that develo objects in a more easy way and can also express the objects. Today, first-order reasoning is a fundament reasoning for machine learning systems in artificial systems all use first- or higher-order logic, which all reasoning and inference in an automated manner.

The basic fact of First-order logic is that first-order logic, but the opposite isn't true. This means tha propositional logic can be treated under first-orde in first-order logic can be treated in propositional l propositions and logical operators in order to consti as units of analysis. First-order logic, in addition quantifiers, objects and relationships.

When an interpretation is given by a symbo formed formula have the meaning. We are int formed formulas whose symbols come from diffe languages, a statement in mathematical theory is a definite syntactical construction. Let Ł be a fir simply called language).

Unlike natural languages, a statement in ma symbolically having an unambiguous syntactica order language. Then, Ł consists of two types of s and non-logical symbols. Logical symbols con $(x_0, x_1, x_2, \ldots)$ or, $(x, y, z, u, \ldots)$, logical connective $(\exists, \forall)$ and the equality symbol $(=)$. Logical sym only interpreted in the sense of the logical op their meaning is never conditioned by the doma first-order logic. On the other hand, non-logical relationships, but also constants and functions. first order language Ł consists of a set of const of n-ary function symbols $\{f_j : j \in J_n\}$ for each p relation symbols $\{R_k : k \in K_n\}$. The meaning asso

3.7 First order Logic

First-order logic can be understood as an extension of propositional logic. In propositional logic the atomic formulas have no internal structure—they are propositional variables that are either true or false. In first-order logic the atomic formulas are predicates that assert a relationship among certain elements. A significant concept in first-order logic is quantification: the ability to assert that a certain property holds for all elements or that it holds for some element.

First-order logic is also known as predicate logic or First-order predicate logic. First-order logic is a powerful language that develops information about the objects in a more easy way and can also express the relationship between those objects. Today, first-order reasoning is a fundamental component of symbolic reasoning for machine learning systems in artificial intelligence. Modern expert systems all use first- or higher-order logic, which allows the conduct of abstract reasoning and inference in an automated manner.

The basic fact of First-order logic is that first-order logic includes propositional logic, but the opposite isn't true. This means that all problems expressed in propositional logic can be treated under first-order logic, but not all problems in first-order logic can be treated in propositional logic. Propositional logic uses propositions and logical operators in order to constitute its well-formed formulas as units of analysis. First-order logic, in addition to those, also uses variables, quantifiers, objects and relationships.

When an interpretation is given by a symbol then we can say that well-formed formula have the meaning. We are interested in interpreting well-formed formulas whose symbols come from different languages. Unlike natural languages, a statement in mathematical theory is expressed as a symbol having a definite syntactical construction. Let Ł be a first order language (that is also simply called language).

Unlike natural languages, a statement in mathematical theory is expressed symbolically having an unambiguous syntactical construction. Let Ł be a first order language. Then, Ł consists of two types of symbols, namely, logical symbols and non-logical symbols. Logical symbols consist of a sequence of variables $\{x_0, x_1, x_2, ...\}$ or, $\{x, y, z, u, ...\}$, logical connectives $(\neg, \wedge, \vee, \rightarrow)$, logical quantifiers $(\exists, \forall)$ and the equality symbol $(=)$. Logical symbols of this type are always and only interpreted in the sense of the logical operation that they represent, and their meaning is never conditioned by the domain of discussion in which we use first-order logic. On the other hand, non-logical symbols comprise predicates and relationships, but also constants and functions. That is non-logical symbols of the first order language Ł consists of a set of constants $\{c_j : i \in I\}$ or $\{a, b, c, ...\}$, a set of n-ary function symbols $\{f_j : j \in J_n\}$ for each positive integer n and a set of n-ary relation symbols $\{R_k : k \in K_n\}$. The meaning associated with non-logical symbols is

domain-specific, and their conversion to sentences in a natural language
conversion rules and interpretation.

We can express a first order language L using the following symbols :

(a) The propositional connectives – and → and the universal
symbol ∀.

(b) Punctuation marks, namely, the left parenthesis "(", the right
")" and the comma ",".

(c) Denumerable collection of individual variables $x_1, x_2, \ldots$

(d) A finite or denumerable or empty set of individual constants.

(e) A non-empty set of predicate variables.

The primitive symbols of a first-order language are the logical symbols
below :

(1) The propositional connectives : $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$.

(2) The quantifiers : $\forall, \exists$

(3) The equality symbol : =

(4) Variables : $x, y, z, x_1, x_2, \ldots$

(5) Parentheses : (,).

The non-logical symbols :

(a) Constant symbols : *a, b, c, ...* (Albert Einstein, numbers, India,
etc.).

(b) Predicate symbols : P, Q, R, ... (>, Brother, red, prime, etc.)

(c) Function symbols: *f, g, h, ...* (Sqrt, father of, friend, etc.)

A language for arithmetic might contain function variables for addition
multiplication and a predicate variabler for equality, whereas a language
geometry is likely to have predicate variable for equality and the notions of
and *line* but no function variables at all.

3.7.1 Syntax of First-Order Logic

The syntax of first-order logic is defined relative to a *signature*. A signature
σ consists of a set of constant symbols, a set of function symbols and a set
predicate symbols. Each function and predicate symbol has an arity
will often refer to predicates as relations. We use letters *c, d* to denote
symbols, *f, g* to denote function symbols and P, Q, R to denote predicate
We must consider that the elements of a signature are symbols; only later
interpret them as concrete functions or relations. Independent of the signature
we also have an infinite set of variables $x_0, x_1, x_2, \ldots$.

Given a signature σ, the set of σ-terms is defined by the following
process :

- Each variable is a term.
- Each constant symbol is a term.

- If $t_1, \ldots, t_k$ are terms and f is a k-ary function symbol
- Given terms $t_1, \ldots, t_k$ and a k-ary predicate symbol P
- For each formula F, –F is a formula.
- For each pair of formulas F, G, (F ∨ G) and (F ∧ G)
- If F is a formula and x is a variable then $\exists x$ F and $\forall$

atomic formulas are those constructed according to the
atomic formula $P(t_1, \ldots, t_k)$ is read "$t_1, \ldots, t_k$ are in relatio
called the existential quantifier. The formula $\exists x$ H is read "
symbol ∀ is called the universal quantifier. The formula ∀
general first-order formula is built up from atomic for
connectives and the two quantifiers. If a formula F occ
formula G then F is called a subformula of G. We assu
bind more tightly than any of the logical operators, e.g
formula (∀x F) ∧ G.

The following are some properties on subformulas

(a) The subformulas of –F are all the subformulas
itself.

(b) If F be an atomic formula, F is itself a subformul

(c) Let ◻ represents any one of the connectives
subformulas of $F_1 \square F_2$ are all the subformulas
$F_1 \square F_2$ itself.

(d) The subformulas of $\forall x$ F, $\exists x$ F are all the subform
$\forall x$ F, $\exists x$ F themselves.

Example 3.18 : Consider a signature with a singl
Since there are no constant symbols or function
variables. An example of a formula is $\forall x \forall y \forall z$
This formula expresses that R is a transitive relat

Example 3.19 : Consider a signature with a consta
symbol +, and a binary relation symbol < $(y + 1)$
Note that many accounts of first-order logic use t
function symbols instead of parameters. This makes
complex. Formulas are built from atomic formulas
quantifiers. Every atomic formula is a formula. If A
a variable then (A), ¬A, A ∧ B, A ∨ B, A → B, $(\forall x)$A,
formulas are formulas without parameters. Close
formulas without free variables. The formula of fi
of symbols that are built using certain rules. A

- If $t_1, \ldots, t_k$ are terms and f is a k-ary function symbol then $f(t_1, \ldots, t_k)$ is a term. Next the set of formulas is defined inductively as follows :
  - Given terms $t_1, \ldots, t_k$ and a k-ary predicate symbol P then $P(t_1, \ldots, t_k)$ is a formula.
  - For each formula F, $\neg F$ is a formula.
  - For each pair of formulas F, G, $(F \vee G)$ and $(F \wedge G)$ are both formulas.
  - If F is a formula and x is a variable then $\exists x\, F$ and $\forall x\, F$ are both formulas.

Atomic formulas are those constructed according to the rule given above. The atomic formula $P(t_1, \ldots, t_k)$ is read "$t_1, \ldots, t_k$ are in relation P". The symbol $\exists$ is called the existential quantifier. The formula $\exists x\, H$ is read "there exists x, H". The symbol $\forall$ is called the universal quantifier. The formula $\forall x\, H$ is read "for all x, H". A general first-order formula is built up from atomic formulas using the logical connectives and the two quantifiers. If a formula F occurs as part of another formula G then F is called a subformula of G. We assume that the quantifiers bind more tightly than any of the logical operators, e.g., $\forall x\, F \wedge G$ denotes the formula $(\forall x\, F) \wedge G$.

The following are some properties on subformulas in predicate logic

(a) The subformulas of $\neg F$ are all the subformulas of F and the formula $\neg F$ itself.

(b) If F be an atomic formula, F is itself a subformula.

(c) Let $\square$ represents any one of the connectives $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$. Then the subformulas of $F_1 \square F_2$ are all the subformulas of F_1, F_2 and the formula $F_1 \square F_2$ itself.

(d) The subformulas of $\forall x\, F$, $\exists x\, F$ are all the subformulas of F and the formulas $\forall x\, F$, $\exists x\, F$ themselves.

Example 3.18 : Consider a signature with a single binary relation symbol R. Since there are no constant symbols or function symbols, the only terms are variables. An example of a formula is $\forall x\, \forall y\, \forall z\, (R(x, y) \wedge R(y, z) \rightarrow R(x, z))$. This formula expresses that R is a transitive relation.

Example 3.19 : Consider a signature with a constant symbol 1, binary function symbol +, and a binary relation symbol $<$. $(y + 1)$ is a formula.

Note that many accounts of first-order logic use terms built from variables and function symbols instead of parameters. This makes the formal details a bit more complex. Formulas are built from atomic formulas using logical connectives and quantifiers. Every atomic formula is a formula. If A and B are formulas and x is a variable then (A), $\neg$A, A $\wedge$ B, A $\vee$ B, A $\rightarrow$ B, $(\forall x)$A, and $(\exists x)$A are formulas. Pure formulas are formulas without parameters. Closed formulas (or sentences) are formulas without free variables. The formula of first order language are strings of symbols that are built using certain rules. A σ-formula is a formula with

non-logical symbols from σ. Logical symbols are unaltered in all languages.
order to specify a language, we have to specify only non-logical symbols. A
language L has finitely many non-logical symbols. A finite sequence of symbols
of a language L is called an expression ∈ in L.

The set of all formulas of L is the smallest set of all expressions of L which
contains all atomic formulas and that is closed under ¬, ∨, ∃.

Now, let us consider the characteristics of different algebraic languages
following examples :

Example 3.20 : The formal language of set theory is the first-order language
whose only non-logical symbol is ∈ (belongs to). Thus, the language for set
theory is finite and countable. Thus, σ = {∈}.

Example 3.21 : The formal language of group theory is the first-order language
whose only non-logical symbols are e (identity element of the group) and
(binary operation). Thus, σ = {e, o}, Thus, the language for group theory
finite as the vocabulary of strings are countable.

Example 3.22 : The formal language of field theory is also first-order language
which is finite and countable as the vocabulary of strings here are the non-
logical symbols 0 (additive identity of ring), 1 (multiplicative identity of ring
'+' and '.' (binary operators). 0 and 1 are constant symbols and '+' and '.' are
binary function symbols. Thus, σ = {0, 1, +, .}.

Example 3.23 : The language of arithmetic is also first-order language which
is finite and countable as the vocabulary of strings are 0, 1, + (addition
constant), (multiplication constant) and ≤ (binary relation symbol).
Thus, σ = {0, 1,+, ., ≤}.

3.7.2 Free and Bound variables

The scope of a quantifier is the smallest formula that follows the quantifier.
This means the part of the logical expression or predicate formula to which
quantifier is applied is called the scope of the quantifier. In $(\forall x)Px \vee Qx$ the scope
of $(\forall x)$ is just Px, while Qx is outside the scope of the quantifier. To include
in the scope of $(\forall x)$ one has to add parentheses: $(\forall x)(Px \vee Qx)$. Note that
conventions about the scope of quantifiers differ in the literature. A variable
occurs bound in a formula A if it occurs in the scope of a quantifier. Any other
occurrence of x in A is free. In other words, when a quantifier is used on a variable
x or when we have to assign a value to this variable to get a proposition, the
occurrence of the variable is said to be *bound* or the variable is said to be
variable. An occurrence of a variable which is not bound by a quantifier or that
set equal to a particular value is said to be free.

In the following table some examples of bound and free var

Table 3.1

Sl. No.	Predicate formula	Bound variable and scop
1.	$\forall x P(x, y)$	x: $P(x, y)$
2.	$\forall x (P(x) \rightarrow Q(x))$	x: $P(x) \rightarrow Q(x)$
3.	$\forall x (P(x) \rightarrow E(y) Q(x, y))$	x: $P(x) \rightarrow E(y) Q(x, y)$ y: $Q(x, y)$
4.	$\forall x (P(x) \wedge Q(x)) \vee \forall y R(y)$	x: $P(x) \wedge Q(x)$
5.	$\exists x P(x) \wedge Q(x)$	First x : $P(x)$ Second x

3.7.3 Semantics of First-Order Logic

An interpretation of first-order logic consists of a no
mappings for function and predicate symbols. Every
is mapped to a function from D^n to D, and every n-p
mapped to a function from Dn to the set comprised of

The domain D is the range of all variables in formul
is called the *domain of the interpretation.*

For a given interpretation, the truth table of any
following rules.

1. The truth tables for propositional connectives ap
   $F \wedge G$, $F \vee G$, $F \rightarrow G$, $\neg F$.
2. $\forall xF$, is true if F is true for any element of D as v
   of x in F. Otherwise, $\forall xF$, is false.
3. $\exists xF$, is true if F is true for at least one eleme
   occurrences of x in F. Otherwise, $\exists xF$, is false.

Truth tables for infinite domains of interpretati
of first-order logic that are tautologies in any in
formulas. A formula is called *satisfiable* if it tak
some interpretation. A formula whose truth tab
interpretation is called *unsatisfiable.*

We can interpret constant, function, and r
way. An interpretation of constant symbol c in
An interpretation of a function symbol f with ar
elements of D to another element of D. An interp
with arity n is the set of n tuples of elements of D

It is important to emphasize the difference
symbol (or function symbol, or constant symbol
function, or object) to which it is interpreted. Th
other symbols, and has no meaning on its own u
Strictly speaking, it makes no sense to write
predicate symbol and 3 is a natural number,

In the following table some examples of bound and free variables are shown :

Table 3.1

Sl. No.	Predicate formula	Bound variable and scope	Free variable
1.	$\forall x P(x, y)$	x: $P(x, y)$	y
2.	$\forall x (P(x) \rightarrow Q(x))$	x: $P(x) \rightarrow Q(x)$	–
3.	$\forall x (P(x) \rightarrow E(y) Q(x, y))$	x: $P(x) \rightarrow E(y) Q(x, y)$ y: $Q(x, y)$	–
4.	$\forall x (P(x) \wedge Q(x)) \vee \forall y R(y)$	x: $P(x) \wedge Q(x)$	–
5.	$\exists x P(x) \wedge Q(x)$	First x : $P(x)$ Second x	

3.7.3 Semantics of First-Order Logic

An interpretation of first-order logic consists of a non-empty domain D and mappings for function and predicate symbols. Every n-place function symbol is mapped to a function from D^n to D, and every n-place predicate symbol is mapped to a function from Dn to the set comprised of two values true and false.

The domain D is the range of all variables in formulas of first-order logic, and is called the *domain of the interpretation*.

For a given interpretation, the truth table of any formula is defined by the following rules.

1. The truth tables for propositional connectives apply to evaluate the value of $F \wedge G$, $F \vee G$, $F \rightarrow G$, $-F$.
2. $\forall xF$, is true if F is true for any element of D as value of x at free occurrences of x in F. Otherwise, $\forall xF$, is false.
3. $\exists xF$, is true if F is true for at least one element of D as value of x at free occurrences of x in F. Otherwise, $\exists xF$, is false.

Truth tables for infinite domains of interpretation are infinite. The formulas of first-order logic that are tautologies in any interpretation are called *valid formulas*. A formula is called *satisfiable* if it takes at least one true value in some interpretation. A formula whose truth table contains only false in any interpretation is called *unsatisfiable*.

We can interpret constant, function, and relation symbols in a similar way. An interpretation of constant symbol c in domain D is an element of D. An interpretation of a function symbol f with arity n is a function that maps n elements of D to another element of D. An interpretation of a relation symbol R with arity n is the set of n tuples of elements of D for which R is true.

It is important to emphasize the difference between a syntactic predicate symbol (or function symbol, or constant symbol) and the semantic predicate (or function, or object) to which it is interpreted. The former is a symbol, relates to other symbols, and has no meaning on its own until we specify an interpretation. Strictly speaking, it makes no sense to write *prime* (say 3) where *prime* is a predicate symbol and 3 is a natural number, since the argument to *prime* is

supposed to be a syntactic term. Sometimes we may obscure this distinc... as above when we specified a language with constant symbols 0, 1, and 2... there is still a fundamental distinction between the objects of the domain and ... symbols we use to represent them.

Sometimes, when we interpret a language in a particular domain, it is useful ... implicitly introduce new constant symbols into the language to denote elements ... of this domain. Specifically, for each element a of the domain, we introduce ... a constant symbol $\bar{a}$, which is interpreted as a. Then the expression *prime* ... $(\bar{3})$ does make sense. Interpreting the predicate symbol *prime* in the natural ... this expression will evaluate to true. We think of $\bar{3}$ as a linguistic "name" ... represents the natural number 3.

3.7.4 Truth in a Model

In a first-order language, suppose we have chosen a domain D in which ... interpret the language, along with an interpretation in D of each of the symbols ... that language. This structure is called the domain D, paired with the interpretation ... called a model for the language. A model for a first-order language is direct ... analogous to a truth assignment for propositional logic, because it provides ... the information we need to determine the truth value of each sentence in the ... language.

The interpretation of a term in a model is an element of the domain of the ... model. The model directly specifies how to interpret constant symbols. To ... interpret a term $f(t)$ created by applying a function symbol to another term, ... we interpret the term t, and then apply the interpretation of f to this term. The ... process makes sense, since the interpretation of f is a function on the domain. ... This generalizes to functions of higher arity in the obvious way. We will not ... interpret terms that include free variables like x and y, since these terms do not ... pick out unique elements of the domain. The variable x could potentially refer to ... any object.

For example, suppose we have a language with two constant symbols, a and ... a unary function symbol f, and a binary function symbol g. Let M be the model ... with domain N (set of natural number), where a and b are interpreted as 3 and ... respectively, $f(x)$ is interpreted as the function which maps any natural number ... to n^2 and g is the addition function. Then the term $g(f(a), b)$ $g(f(a), b)$ denotes the ... natural number $3^2 + 5 = 14$.

Similarly, the interpretation of an assertion is a value T or F. For the sake of ... brevity, we will introduce new notation here: if A is an assertion and M is a model ... of the language of A, we write $M \models A$ to mean that A evaluates to T in M, and ... $M \not\models A$ to mean that A evaluates to F. We read the symbol $\models$ as "*models*" or ... "*satisfies*" or "*validates*".

To interpret a predicate or relation applied to some terms, we first interpret ... the terms as objects in the domain, and then see if the interpretation of the ...

relation symbol is true of those objects. To continue with the ... our language also has a relation symbol R, and we extend M to ... greater-than-or-equal-to relation. Then we have $M \models R(a, b)$, since ... than 5, but $M \models R(g(f(a), b), b)$, since 14 is greater than 5.

Interpreting expressions using the logical connectives ... exactly as it did in the propositional setting. $M \models A \wedge B$ exactly when ... $M \models B$, and so on.

We still need to explain how to interpret existential and universal ... We saw that $\exists x A$ intuitively meant that there was some element ... that would make A true, when we "replaced" the variable x with ... To make this a bit more precise, we say that $M \models \exists x A$ exactly when ... element a in the domain of M such that, when we interpret x as ... To continue the example above, we have $M \models \exists x(R(x, b))$, since when ... x as 6 we have $M \models R(x, b)$.

More concisely, we can say that $M \models \exists x A$ when there is an a ... M such that $M \models A[\bar{a}\,/x]$. The notation $A[\bar{a}\,/x]$ indicates that every ... x in A has been replaced by the symbol $\bar{a}$.

Finally we find that $\forall x A$ means that A is true for all possible ... make this precise by saying that $M \models \forall x A$ exactly when for every ... domain of M, interpreting x as a gives that $M \models A$. Alternatively, ... $M \models \forall x A$ when for every a in the domain of M, we have ... example above, $M \not\models \forall x(R(x, b))$, since when we interpret x ... $M \models R(x, b)$. These rules allow us to determine the truth value ... a model. We must remember that a *sentence* is a formula with ...

Let us consider a simple language with no constant ... symbol $\leq$, and one binary function symbol +. Our model N ... and the symbols will be interpreted as the standard less-than ... and addition function. Now, we shall consider the following ... domain is N, not Z or any other number system:

Example 3.24 : We consider the truth of $M \models \exists x(x \leq$ ...

Here, we find that both of these statements are true ... interpret x as the natural number 0. Then, $M \models x \leq x$. ... For the latter, let us consider an arbitrary natural number ... that when we interpret x as n, we have $M \models x \leq x$.

Example 3.25 : We consider the truth of $M \models \exists x(x +$ ...

The first statement is true, since we can interpret x ... is false. When we interpret x as 1 (or, in fact, as any ... we see that $M \not\models x + x \leq x$.

Example 3.26 : The sentences $\exists x\, \forall y\, (x \leq y)$ and ... same meaning and both are true in the specified ... that some whole number is less than or equal to ...

relation symbol is true of those objects. To continue with the example, suppose our language also has a relation symbol R, and we extend M to interpret R as the greater-than-or-equal-to relation. Then we have $M \not\models R(a, b)$, since 3 is not greater than 5, but $M \models R(g(f(a), b), b)M$, since 14 is greater than 5.

Interpreting expressions using the logical connectives $\wedge$, $\vee$, $\rightarrow$, and $\neg$ works exactly as it did in the propositional setting. $M \models A \wedge B$ exactly when $M \models A$ and $M \models B$, and so on.

We still need to explain how to interpret existential and universal expressions. We saw that $\exists xA$ intuitively meant that there was some element of the domain that would make A true, when we "replaced" the variable x with that element. To make this a bit more precise, we say that $M \models \exists xA$ exactly when there is an element a in the domain of M such that, when we interpret x as a, then $M \models A$. To continue the example above, we have $M \models \exists x(R(x, b))$, since when we interpret x as 6 we have $M \models R(x, b)$.

More concisely, we can say that $M \models \exists xA$ when there is an aa in the domain of M such that $M \models A[\bar{a}/x]$. The notation $A[\bar{a}/x]$ indicates that every occurrence of x in A has been replaced by the symbol $\bar{a}$.

Finally we find that $\forall xA$ means that A is true for all possible values of x. We make this precise by saying that $M \models \forall xA$ exactly when for every element a in the domain of M, interpreting x as a gives that $M \models A$. Alternatively, we can say that $M \models \forall xA$ when for every a in the domain of M, we have $M \models A[\bar{a}/x]$. In our example above, $M \not\models \forall x(R(x, b))$, since when we interpret x as 2 we do not have $M \models R(x, b)$. These rules allow us to determine the truth value of any *sentence* in a model. We must remember that a *sentence* is a formula with no free variables.

Let us consider a simple language with no constant symbols, one relation symbol $\leq$, and one binary function symbol $+$. Our model M will have domain N, and the symbols will be interpreted as the standard less-than-or-equal-to relation and addition function. Now, we shall consider the following examples where our domain is N, not Z or any other number system:

Example 3.24 : We consider the truth of $M \models \exists x(x \leq x)$ and $M \models \forall x(x \leq x)$

Here, we find that both of these statements are true. For the former, we can interpret x as the natural number 0. Then, $M \models x \leq x$, so the existential is true. For the latter, let us consider an arbitrary natural number n; it is still the case that when we interpret x as n, we have $M \models x \leq x$.

Example 3.25 : We consider the truth of $M \models \exists x(x + x \leq x)$ and $M \models \forall x(x + x \leq x)$. The first statement is true, since we can interpret x as 0. The second statement is false. When we interpret x as 1 (or, in fact, as any natural number besides 0), we see that $M \not\models x + x \leq x$.

Example 3.26 : The sentences $\exists x \forall y (x \leq y)$ and $\forall x \exists y(x \leq y)$ do not have same meaning and both are true in the specified model. The first expresses that some whole number is less than or equal to every whole number. This

is true : 0 is less than or equal to every whole number. The second sen
says that for every whole number, there is another whole number at le
big. Again, this is true: every whole number a is less than or equal to a
take our domain to be Z instead of whole number, the first sentence
be false, while the second would still be true.

Example 3.27 : It is not possible to think of a formula A in the language
one free variable x, such that $M \models \forall x A$ but $M \not\models \exists x A$. The situation descri
here is impossible in our model. If $M \models \forall x A$, then $M \models A[\bar{0}/x]$ which im
that $M \models \exists x A$. The only time this situation can happen is when the do
of our model is empty.

Now, let us consider a formula F of the first order language L. Then we
some important assumptions as the following

1. The formula F is valid if and only if it is satisfied by every interpretation
2. It will be satisfiable formula if and only if it is satisfied by some interpret
   of L and
3. It will be unsatisfiable if and only if it is not satisfied by any interpreta
   of L.

It can be also told that, an assertion involving predicates is said to be va
if it is true for all elements in the universe of discourse. We can also say tha
compound proposition is satisfiable if there is atleast one true result in the tru
table.Interpretations for the expressions of predicate logic are possible meanin
for the predicates and variables. They are comparable to truth assignments
propositional logic. So, we can say that an assertion which involves predicates
said to be *satisfiable* if there exist a universe of discourse and an *interpretation* h
which the assertion is true. Otherwise, it will be *unsatisfiable*. We can also say th
it will be *unsatisfiable* if there is not even a single true result in the truth table.

A formula F is called *valid* iff it is true in all interpretations, i.e. $I \models F$ for a
interpretations I. Because any interpretation makes valid formulas true, we als
write $\models F$ iff formula F is valid. A formula F is called *satisfiable* iff there is a
interpretation I in which it is true, i.e. $I \models F$. Otherwise it is called *unsatisfiab*
Satisfiability and validity are duals of each other. That is, a formula F is valid
and only if $\sim F$ is *unsatisfiable*.

F is valid $\leftrightarrow \sim F$ is unsatisfiable

Importantly, this means that we can decide whether a formula is valid b
reasoning about the satisfiability of its negation. A proof of validity for F fro
the unsatisfiability of $\sim F$ is called a *refutation*. Most efficient decision procedure
use this approach, and therefore attempt to directly prove the satisfiability of
given formula.

As was the case with propositional logic, we can extend the notion of valid
to a notion of logical consequence. Let us consider a first-order language
Suppose Γ is a set of sentences in L, and A is a sentence of L. We will say that A

a logical consequence of Γ if every model of Γ is a model o
spelling out that A is a "necessary consequence" of A : un
if the hypotheses in Γ come out true, A is true as well.

3.8 Valid Formula and Equivalences

Let A and B be any two predicate formulas defined
of discourse E. When each of the variables appearing
the same truth values, then A and B are said to be equ
E and denoted as $A \equiv B$ or $A \Leftrightarrow B$.

Generally, logically valid formulas in predicate
from tautologies of propositional calculus by repla
(elementary statements) such as p, q, r by propositio

For example, $p \vee \sim p \equiv T$ and $(p \rightarrow q) \leftrightarrow (\sim p \vee q) \equiv T$
calculus.

If we replace p by $\forall x\, R(x)$and q by $\exists x\, S(x)$ in the
valid formulas in predicate calculus.

$(\forall x\, R(x)) \vee (\sim \forall x\, R(x)) \equiv T$

$(\forall x\, R(x)) \rightarrow \exists x\, S(x)) \leftrightarrow ((\sim \forall x\, R(x)) \vee \exists x\, S(x))$

More generally, all the implication and equivale
can also be considered as implications and equiva
if we replace elementary statement by primary pr

from $\sim\sim p \Leftrightarrow p$ we get $\sim\sim P(x) \leftrightarrow P(x)$

from $p \wedge q \equiv q \wedge p$, we get $P(x) \wedge Q(x, y) \equiv Q(x,$

from $p \rightarrow q \equiv \sim p \vee q$, we get $P(x) \rightarrow Q(x) \equiv \sim P($

(1), (2) and (3) are some examples for valid fo

Apart from the types of valid formulas giv
formulas also which involve quantifiers. Such
using the inference theory of predicate logic, di

3.9 Inference Theory of Predi

Derivations of formal proof in predicate cal
way as in statement calculus using implicatio
the statement formulas are replaced by predic
rules P, T and CP of Inference theory used in st
in predicate calculus. Moreover, the induct m
predicate calculus.

Apart from the above rules of inference,
to deal with predicate formulas involving q
to eliminate quantifiers during the course of

a logical consequence of Γ if every model of Γ is a model of *A*. This is one way of spelling out that *A* is a "necessary consequence" of *A* : under any interpretation, if the hypotheses in Γ come out true, *A* is true as well.

3.8 Valid Formula and Equivalences

Let A and B be any two predicate formulas defined over a common universe of discourse E. When each of the variables appearing in A and B is replaced by the same truth values, then A and B are said to be equivalent to each other over E and denoted as $A \equiv B$ or $A \Leftrightarrow B$.

Generally, logically valid formulas in predicate calculus can be obtained from tautologies of propositional calculus by replacing primary propositions (elementary statements) such as p, q, r by propositional functions.

For example, $p \vee \neg p \equiv T$ and $(p \to q) \leftrightarrow (\neg p \vee q) \equiv T$ are tautologies in statement calculus.

If we replace p by $\forall x\, R(x)$and q by $\exists x\, S(x)$ in the above, we get the following valid formulas in predicate calculus.

$(\forall x\, R(x)) \vee (\neg \forall x\, R(x)) \equiv T$

$(\forall x\, R(x)) \to \exists x\, S(x)) \leftrightarrow ((\neg \forall x\, R(x)) \vee \exists x\, S(x)) \equiv T$

More generally, all the implication and equivalence of the statement calculus can also be considered as implications and equivalence of the predicate calculus if we replace elementary statement by primary predicate formula.For example,

from $\neg\neg p \Leftrightarrow p$ we get $\neg\neg P(x) \leftrightarrow P(x)$ ... (1)

from $p \wedge q \equiv q \wedge p$, we get $P(x) \wedge Q(x, y) \equiv Q(x,y) \wedge P(x)$ ... (2)

from $p \to q \equiv \neg p \vee q$, we get $P(x) \to Q(x) \equiv \neg P(x) \vee Q(x)$ ... (3)

(1), (2) and (3) are some examples for valid formulas in predicate calculus.

Apart from the types of valid formulas given above, there are other valid formulas also which involve quantifiers. Such valid formulas are obtained by using the inference theory of predicate logic, discussed below.

3.9 Inference Theory of Predicate Calculus

Derivations of formal proof in predicate calculus are done mostly in the same way as in statement calculus using implications and equivalences, provided that the statement formulas are replaced by predicate formulas. Also the three basic rules P, T and CP of Inference theory used in statement calculus can also be used in predicate calculus. Moreover, the induct method of proof can also be used in predicate calculus.

Apart from the above rules of inference, we require certain additional rules to deal with predicate formulas involving quantifiers. If it becomes necessary to eliminate quantifiers during the course of derivation, we require two rules of

specification, called US and ES rules. Once the quantifiers are eliminated, the derivation is similar to that in statement calculus. If it becomes necessary to quantify the desired conclusion, we require two rules of generalisation, called UG and EG rules.

Rule US : Universal Specification is the rule of inference which states that one can conclude that $P(c)$ is true, if $\forall x\, P(x)$ is true, where c is an arbitrary member of the universe of discourse. This rule is also called the universal instantiation.

Rule ES : Existential Specification is the rule which allows us to conclude that $P(c)$ is true, if $\exists x\, P(x)$ is true, where c is not an arbitrary member of the universe but one for which $P(c)$ is true. Usually, we will not know what c is but we know that it exists. Since it exists, we may call it c. This rule is also called the *existential instantiation.*

Rule UG : Universal Generalisation is the rule which states that $\forall x\, P(x)$ is true if $P(c)$ is true, where c is an arbitrary member (not a specific member) of the universe of discourse.

Rule EG : Existential Generalisation is the rule that is used to conclude that $\forall x\, P(x)$ is true when $P(c)$ is true, where c is a particular member of the universe of discourse.

3.10 Formal theory for Predicate Calculus

Formal theory for predicate calculus is given by the following :

(a) Formal language: It is a language of the respective proof calculus (a set Σ of *WFFs*).

(b) A set of axioms : It is a subset of the set Σ of all WFFs that consists of
   (i) a set of logical axioms and (ii) a set of special axioms.

(c) A set of deduction rules : It is the set of rules of respective calculus.

3.11 Theorems and Derivations

A set of deduction rules of the form $A_1, A_2, ..., A_n \mapsto B_1, B_2, ..., B_n$ enables us for proving theorems. The form $A_1, A_2, ..., A_n \mapsto B_1, B_2, ..., B_n$ implies that each B_i, $i = 1, 2, ... n$ is derived from the set of assumptions $A_1, A_2, ..., A_n$. Here we are concerned with three most important theories given below.

(a) *Arithmetic theories* : Peano arithmetic.

(b) *Relation theories* : theories of ordering and equivalence.

(c) *Algebraic theories* : theories of groups, rings, fields, theories of lattices/.

Proof of a formula A in the theory Θ is a sequence of WFFs (proof steps) such that

(i) the final step is the formula A.

(ii) each step of the proof is either a logical axiom or a formula obtained by applying a deduction rule to some of the sequence.

A provable formula φ in the theory Θ is often denoted as $\vdash \varphi$ and is said to be provable from $A_1, A_2, ..., A_n$ i.e. $A_1, A_2, ..., A_n \vdash \varphi$ or φ from the set of assumptions $\{A_1, A_2, ..., A_n\}$. φ is a *theorem.*

A proof system is formed from a set of rules and axioms. The proofs, also called *derivations.* Every derivation has a conclusion which is known as *statement proved* or *statement*

3.12 First Order Theory with

Different conventions are there for using equality in first order logic. The most common convention, known as first order logic includes the equality symbol as a primitive. The axioms for equality interpreted as the real equality relation between elements of the discourse, such that the "two" given may be considered as equal. The equality axioms are :

1. Reflexivity. For each variable x, $x =$

2. Substitution for functions. For a symbol f, we write

   $x = y \rightarrow f(..., x, ...) = f(..., y, ...$

3. Substitution for formulas. For any φ' is obtained by replacing any n such that these remain free occu

   then $x = y \rightarrow (\varphi \rightarrow \varphi')$.

These are axiom schemas, each of in mathematical logic, the axiom s The third schema is known as Leibn function symbol f, is (equivalent to) formula

$x = y \rightarrow (f(..., x, ...) = z \rightarrow f(...$

Many other properties of equali example :

1. **Symmetry :** If $x = y$ then $y = x$ being $x = x$ and φ' being $y =$

2. **Transitivity :** If $x = y$ and $y = z$ with φ being $y = z$ and φ' bei we use the reflexive propert

(ii) each step of the proof is either a logical axiom, or a special axiom, or a formula obtained by applying a deduction rule to some previous formulas of the sequence.

A provable formula φ in the theory Θ is often denoted by $\Theta \mapsto \varphi$. A formula φ is said to be provable from $A_1, A_2, ..., A_n$ i.e. $A_1, A_2, ..., A_n \mapsto \varphi$, if there is a proof of φ from the set of assumptions $\{A_1, A_2, ..., A_n\}$. A provable formula is known as a *theorem*.

A proof system is formed from a set of rules that chained together to form proofs, also called *derivations*. Every derivation has exactly one final conclusion, which is known as *statement proved* or *statement derived*.

3.12 First Order Theory with Equality

Different conventions are there for using equality (or identity) in first-order logic. The most common convention, known as first-order logic with equality, includes the equality symbol as a primitive logical symbol which is always interpreted as the real equality relation between members of the domain of discourse, such that the "two" given members are the same member. These equality axioms are :

1. Reflexivity. For each variable x, $x = x$.
2. Substitution for functions. For all variables x and y, and any function symbol f, we write
   $x = y \rightarrow f(..., x, ...) = f(..., y, ...)$.
3. Substitution for formulas. For any variables x and y and any formula $\varphi(x)$, if φ' is obtained by replacing any number of free occurrences of x in φ with y, such that these remain free occurrences of y,
   then $x = y \rightarrow (\varphi \rightarrow \varphi')$.

These are axiom schemas, each of which specifies an infinite set of axioms. In mathematical logic, the axiom schemas generalises the notion of axioms. The third schema is known as Leibniz's law. The second schema, involving the function symbol f, is (equivalent to) a special case of the third schema, using the formula

$x = y \rightarrow (f(..., x, ...) = z \rightarrow f(..., y, ...) = z)$.

Many other properties of equality are consequences of the axioms above, for example :

1. **Symmetry :** If $x = y$ then $y = x$. We must use the substitution formula with φ being $x = x$ and φ' being $y = x$ and, then we use the reflexive property
2. **Transitivity :** If $x = y$ and $y = z$ then $x = z$. We must use the substitution formula with φ being $y = z$ and φ' being $x = z$ to obtain $y = x \rightarrow (y = z \rightarrow x = z)$ and, then we use the reflexive property

Note

An alternate approach considers the equality relation to be a non-l... symbol. This convention is known as *first-order logic without equality*. ... equality relation is included in the signature, the axioms of equality ... now be added to the theories under consideration, if desired, instead of ... considered as rules of logic. The main difference between this method ... first-order logic with equality is that an interpretation may now interpret ... distinct individuals as "equal" (although, by Leibniz's law, these will sat... exactly the same formulas under any interpretation). That is, the equa... relation may now be interpreted by an arbitrary equivalence relation on ... domain of discourse that is congruent with respect to the functions ... relations of the interpretation.

3.12.1 Examples of first order theory on Group, Ring and Fie...

1. **Elementary theory of groups :** We first consider group theory. Let x, y, z de... the first three variables in the domain G (group). The proper axioms are t... following.
   (i) $\forall x \forall y \forall z\,(x + (y + z) = (x + y) + z)$
   (ii) $\forall x\,((x + 0 = x) \wedge (0 + x = x))$
   (iii) $\forall x \exists y\,((x + y = 0) \wedge (y + x = 0)$
   If we add to the axioms the following :
   (iv) $\forall x \forall y(x + y = y + x)$
   the new theory is called the *elementary theory of abelian groups.*
2. **Elementary theory of fields :** Let x, y and z express as the first three variable... The proper axioms are the following.
   (i) $\forall x \forall y \forall z\,(x + (y + z) = (x + y) + z)$
   (ii) $\forall x((x + 0 = x) \wedge (0 + x = x))$
   (iii) $\forall x \exists y\,(x + y = 0) \wedge (y + x = 0)$
   (iv) $\forall x \forall y(x + y = y + x)$
   (v) $\forall x \forall y \forall z\,(x.(y.z) = (x.y).z)$
   (vi) $\forall x(x.1 = x \wedge 1.x = x)$
   (vii) $\forall x \forall y \forall z(x.(y + z) = x.y + x.z)$
   (viii) $\forall x \forall y \forall z\,((y + z).x = y.x + z.x)$
   (ix) $\forall x \forall y(x.y = y.x)$
   (x) $\forall x(\neg(x = 0) \rightarrow \exists y(x.y = 1 \wedge y.x = 1))$
3. **Elementary theory of linearly ordered sets :** Let x, y and z denote the first three variables. Let L be a first order language for the theory of linearly ordered sets having only one non-logical symbol '<', a binary relation symbol. The proper axioms are the following.
   (i) $\forall x \neg(x < x)$
   (ii) $\forall x \forall y \forall z\,((x < y \wedge y < z) \rightarrow (x < z))$
   (iii) $\forall x \forall y\,(x < y \vee x = y \vee y < x)$
4. **Elementary theory of ordered field...** variables. The proper axioms are the...
   (i) $\forall x \forall y \forall z(x + (y + z) = (x + y) + z)$
   (ii) $\forall x((x + 0 = x) \wedge (0 + x = x))$
   (iii) $\forall x \exists y((x + y = 0) \wedge (y + x = 0))$
   (iv) $\forall x \forall y(x + y = y + z)$
   (v) $\forall x \forall y \forall z(x.(y.z) = (x.y).z)$
   (vi) $\forall x(x.1 = x \wedge 1.x = x)$
   (vii) $\forall x \forall y \forall z(x.(y + z) = x.y + x.z)$
   (viii) $\forall x \forall y \forall z((y + z).x = y.x + z.x)$
   (ix) $\forall x \forall y\,(x.y = y.x)$
   (x) $\forall x(\neg(x = 0) \rightarrow \exists y(x.y = 1 \wedge y.x$
   (xi) $\forall x \neg(x < x)$
   (xii) $\forall x \forall y \forall z((x < y \wedge y < z) \rightarrow x <$
   (xiii) $\forall x \forall y(x < y \vee x = y \vee y < x)$
   (xiv) $\forall x \forall y(\neg(x < y \vee x = y) \rightarrow y <$
   (xv) $\forall x \forall y(x < y \rightarrow \forall z(x + z < y +$
   (xvi) $\forall x \forall y((0 < x \wedge 0 < y) \rightarrow 0 <$

3.13 Prenex Normal F...

Prenex Normal form is a formu... of the form

$$(Q_1 x$$

where Q_i are either the universal... and P is free of quantifiers that is...

Thus, a first order formula ... quantifiers and bound variables ... be in prenex normal form (PNF) ... is called the *prefix* and the formu...

Example 3.28 : For example... $\exists x \forall y(p(x) \rightarrow q(y))$ is in P... $P = (p(x) \rightarrow q(y))$

Example 3.29 : Another exa...
Here the prefix is $\forall x \exists y \forall z$...
We observe that the form... equivalent to the formula

(i) $\forall x \neg(x < x)$
(ii) $\forall x \forall y \forall z ((x < y \wedge y < z) \rightarrow (x < z))$
(iii) $\forall x \forall y (x < y \vee x = y \vee y < x)$

4. **Elementary theory of ordered fields :** Let x, y and z denote the first three variables. The proper axioms are the following.
(i) $\forall x \forall y \forall z(x + (y + z) = (x + y) + z)$
(ii) $\forall x((x + 0 = x) \wedge (0 + x = x))$
(iii) $\forall x \exists y((x + y = 0) \wedge (y + x = 0))$
(iv) $\forall x \forall y(x + y = y + z)$
(v) $\forall x \forall y \forall z(x.(y.z) = (x.y).z)$
(vi) $\forall x(x.1 = x \wedge 1.x = x)$
(vii) $\forall x \forall y \forall z(x.(y + z) = x.y + x.z)$
(viii) $\forall x \forall y \forall z((y + z).x = y.x + z.x)$
(ix) $\forall x \forall y (x.y = y.x)$
(x) $\forall x(\neg(x = 0) \rightarrow \exists y(x.y = 1 \wedge y.x = 1))$
(xi) $\forall x \neg(x < x)$
(xii) $\forall x \forall y \forall z((x < y \wedge y < z) \rightarrow x < z)$
(xiii) $\forall x \forall y(x < y \vee x = y \vee y < x)$
(xiv) $\forall x \forall y(\neg(x < y \vee x = y) \rightarrow y < x)$
(xv) $\forall x \forall y(x < y \rightarrow \forall z(x + z < y + z))$
(xvi) $\forall x \forall y((0 < x \wedge 0 < y) \rightarrow 0 < x.y)$

3.13 Prenex Normal Form

Prenex Normal form is a formula containing no quantifiers at all, or a formula of the form

$$(Q_1\ x_1)(Q_2\ x_2)\ldots\ (Q_n\ x_n)P$$

where Q_i are either the universal or existential quantifier, x_i are bound variables and P is free of quantifiers that is an open formula.

Thus, a first order formula of predicate calculus containing a string of quantifiers and bound variables followed by a quantifier free formula is said to be in prenex normal form (PNF). The string of quantifiers $(Q_1\ x_1)(Q_2\ x_2)\ldots(Q_n\ x_n)$ is called the *prefix* and the formula is called the *matrix* of the prenex normal form.

Example 3.28 : For example
$\exists x \forall y(p(x) \rightarrow q(y))$ is in PNF. Here the *prefix* is $\exists x \forall y$ and the *matrix* is $P = (p(x) \rightarrow q(y))$

Example 3.29 : Another example of PNF is $\forall x \exists y \forall z(p(y) \vee (q(z) \rightarrow r(x)))$
Here the prefix is $\forall x \exists y \forall z$ and the *matrix* is $P = p(y) \vee (q(z) \rightarrow r(x))$
We observe that the formula $\forall x((\exists y(p(y)) \vee ((\exists z(q(z) \rightarrow r(x)))$ is logically equivalent to the formula $\forall x \exists y \forall z(p(y) \vee (q(z) \rightarrow r(x)))$ but it is not in PNF.

Example 3.30 : $\forall x\,\exists y(x > 0 \rightarrow (y > 0 \wedge x = y^2))$ is in prenex form, $\exists x(x = 0) \wedge \exists y(y < 0)$ and $\forall x(x > 0 \vee \exists y(y > 0 \wedge x = y^2))$ are not in prenex

It is to be noted that in

$$(Q_1\, x_1)(Q_2\, x_2) \ldots (Q_n\, x_n)P$$

if P is in DNF, then it is known as *prenex disjunctive normal form* (PDNF). On the other hand if P is in CNF, then

$$(Q_1\, x_1)(Q_2\, x_2) \ldots (Q_n\, x_n)P$$

is called *prenex conjuctive normal form* (PCNF).

Example 3.31 : $\forall x\,\exists y(\neg x > 0 \vee y > 0)$ is both in PDNF and in PCNF.

$\forall x\,\exists y(\neg x > 0 \vee (y > 0 \wedge \neg x = y^2))$ is in PDNF, but not in PCNF.

$\forall x(x > 0 \vee \exists y(y > 0 \wedge x = y^2))$ is neither in PCNF nor in PDNF.

Let us now learn how to use the concept of Conversion to Prenex Normal f

3.13.1 Conversion to Prenex Normal Form : An Algorithm

There are several rules that can be recursively applied to a first order form in order to reduce it into a PNF. This depends on the logical connectives appear in the formula. Any expression can be converted into prenex norm form. To do this, the following steps are needed :

1. Eliminate all occurrences of $\rightarrow$ and $\leftrightarrow$ from the formula in question. Th replace implications, biconditionals, etc.
   For example $(A \rightarrow B)$ by $(\neg A \vee B)$ and $A \leftrightarrow B \equiv (A \rightarrow B) \wedge (B \rightarrow A)$
2. Move all negations inward such that, in the end, egations only appe part of literals. That is move – "inwards" until there are no quantifiers a scope of a negation, by De Morgan's laws.
3. Standardize the variables apart (when necessary). That is rename varia so that each variable following a quantifier has a unique name. We may the equivalences
   (a) $\forall x P \wedge \forall x Q \equiv \forall x(P \wedge Q)$,
   (b) $\exists x P \vee \exists x Q \equiv \exists x(P \vee Q)$,
4. The prenex normal form can now be obtained by moving all quantifi to the front of the formula. That is move quantifiers to the front of sentence,without changing their order. We may use the equivalences
   (c) $\forall x P \wedge Q \equiv Q \wedge \forall x P \equiv \forall x(P \wedge Q)$,
   (d) $\forall x P \vee Q \equiv Q \vee \forall x P \equiv \forall x(P \vee Q)$,
   (e) $\exists x P \vee Q \equiv Q \vee \exists x P \equiv \exists x(P \vee Q)$,
   (f) $\exists x P \wedge Q \equiv Q \wedge \exists x P \equiv \exists x(P \wedge Q)$,
5. Finally, transform the matrix in a DNF or CNF, just like a propositional form

Note

Every first-order formula is equivalent t *normal form* (PDNF) and to a formula i (PCNF).

Example 3.32 : Let us consider the fi $A = \exists z(\exists x Q(x, z) \vee \exists x P(x)) \rightarrow \neg(\neg \exists x$ conversion to PNF in the following s

1. Eliminating $\rightarrow$: $A \equiv \neg\exists z(\exists x Q(x, z)$
2. Importing the negation : $A \equiv \forall z(\neg$ $Q(z, x)) \equiv \forall z(\forall x \neg Q(x, z) \wedge \forall x \neg P$
3. Using the equivalences (a) and ($A \equiv \forall z \forall x$
4. Renaming : $A \equiv \forall z \forall x(\neg Q(x, z) \wedge$
5. Using the equivalences (c)-(f) t $A \equiv \forall z \forall x \exists y \forall u((\neg Q(x, z) \wedge \neg P$
6. The resulting formula is in a distribute the $\vee$ over $\wedge$: $A \equiv \forall z \forall x \exists y \forall u((\neg Q(x, z) \vee P$

Example 3.33 : We shall find the $\forall x(\exists y R(x, y) \wedge \forall y \neg S(x, y) \rightarrow$

According to Step 1, we must eli $\forall x(\neg \exists y R(x, y) \wedge \forall y \neg S(x, y)) \vee$

According to Step 2, we move al $\forall x(\forall y \neg R(x, y) \vee \exists y S(x, y) \vee$

Next, according to Step 3, all va $\forall x(\forall y_1 \neg R(x, y_1) \vee \exists y_2 S(x, y_2)$

According to Step 4, we can no $\forall x \forall y_1 \exists y_2 \forall y_3(\neg R(x, y_1) \vee S($

Example 3.34 : Example of Co $\forall x((C(x) \wedge \exists y(T(y) \wedge L(x, y))) \rightarrow$ $\forall x((C(x) \wedge \exists y(T(y) \wedge L(x, y))) \vee$ $\forall x(\exists y(C(x) \wedge T(y) \wedge L(x, y)) \vee \exists$ $\forall x \forall y((C(x) \wedge T(y) \wedge L(x, y)) \vee$ $\forall x \forall y \exists z((C(x) \wedge T(y) \wedge L(x, y)$

If we want to restore the impli $\forall x \forall y \exists z(C(x) \wedge T(y) \wedge L$

Another prenex normal form $\forall x \exists z \forall y(C(x) \wedge T(y) \wedge L(x,$

Note

Every first-order formula is equivalent to a formula in a *prenex disjunctive normal form* (PDNF) and to a formula in a *prenex conjunctive normal form* (PCNF).

Example 3.32 : Let us consider the first order formula

$A = \exists z(\exists x Q(x, z) \vee \exists x P(x)) \rightarrow \neg(\neg \exists x P(x) \wedge \forall x \exists z Q(z, x))$. We now show the conversion to PNF in the following steps :

1. Eliminating $\rightarrow$: $A \equiv \neg \exists z(\exists x Q(x, z) \vee \exists x P(x)) \vee \neg(\neg \exists x P(x) \wedge \forall x \exists z Q(z, x))$
2. Importing the negation : $A \equiv \forall z(\neg \exists x Q(x, z) \wedge \neg \exists x P(x)) \vee (\neg\neg \exists x P(x) \vee \neg \forall x \exists z Q(z, x)) \equiv \forall z(\forall x \neg Q(x, z) \wedge \forall x \neg P(x)) \vee (\exists x P(x) \vee \exists x \forall z \neg Q(z, x))$.
3. Using the equivalences (a) and (b) :
$$A \equiv \forall z \forall x(\neg Q(x, z) \wedge \neg P(x)) \vee \exists x(P(x) \vee \forall z \neg Q(z, x)).$$
4. Renaming : $A \equiv \forall z \forall x(\neg Q(x, z) \wedge \neg P(x)) \vee \exists y(P(y) \vee \forall w \neg Q(w, y))$.
5. Using the equivalences (c)-(f) to pull the quantifiers in front:
$A \equiv \forall z \forall x \exists y \forall w((\neg Q(x, z) \wedge \neg P(x)) \vee P(y) \vee \neg Q(w, y))$.
6. The resulting formula is in a prenex DNF. For a prenex CNF we have to distribute the $\vee$ over $\wedge$:
$A \equiv \forall z \forall x \exists y \forall w((\neg Q(x, z) \vee P(y) \vee \neg Q(w, y)) \wedge (\neg P(x) \vee P(y) \vee \neg Q(w, y)))$.

Example 3.33 : We shall find the prenex normal form of

$\forall x(\exists y R(x, y) \wedge \forall y \neg S(x, y) \rightarrow \neg(\exists y R(x, y) \wedge P))$

According to Step 1, we must eliminate $\rightarrow$, which yields

$\forall x(\neg(\exists y R(x, y) \wedge \forall y \neg S(x, y)) \vee \neg(\exists y R(x, y) \wedge P))$

According to Step 2, we move all negations inwards, which yields :

$\forall x(\forall y \neg R(x, y) \vee \exists y S(x, y) \vee \forall y \neg R(x, y) \vee \neg P)$.

Next, according to Step 3, all variables are standardized apart :

$\forall x(\forall y_1 \neg R(x, y_1) \vee \exists y_2 S(x, y_2) \vee \forall y_3 \neg R(x, y_3) \vee \neg P)$

According to Step 4, we can now move all quantifiers in front, which yields

$\forall x \forall y_1 \exists y_2 \forall y_3(\neg R(x, y_1) \vee S(x, y_2) \vee \neg R(x, y_3) \vee \neg P)$.

Example 3.34 : Example of Conversion of Prenex Normal Form of $\forall x((C(x) \wedge \exists y(T(y) \wedge L(x, y))) \rightarrow \exists y(D(y) \wedge B(x, y)))$ is as follows :

$\forall x((C(x) \wedge \exists y(T(y) \wedge L(x, y))) \vee \exists z(D(z) \wedge B(x, z)))$

$\forall x(\exists y(C(x) \wedge T(y) \wedge L(x, y)) \vee \exists z(D(z) \wedge B(x, z)))$

$\forall x \forall y((C(x) \wedge T(y) \wedge L(x, y)) \vee \exists z(D(z) \wedge B(x, z)))$

$\forall x \forall y \exists z((C(x) \wedge T(y) \wedge L(x, y)) \vee (D(z) \wedge B(x, z)))$

If we want to restore the implication then we have

$$\forall x \forall y \exists z(C(x) \wedge T(y) \wedge L(x, y)) \rightarrow (D(z) \wedge B(x, z))$$

Another prenex normal form is :

$$\forall x \exists z \forall y(C(x) \wedge T(y) \wedge L(x, y)) \rightarrow (D(z) \wedge B(x, z))$$

3.14 Choice rule

To semantically justify a WFF in propositional logic, we may use truth table
But to semantically justify a WFF in predicate logic, we must find a model w
consists of a Universe of objects from which the variable x takes values and
correspondence between the logical connectives and operations over the Univers

Generally, every WFF can be proved using a finite number of arbitrary choic
that is known as the choice rule (rule C). In order to introduce the role of choi
rule, let us consider a WFF of the form $\exists x P(x)$. If we would like to prove the WF
$\exists x P(x)$, we say, there exists an object c such that $P(c)$ is true. We continue the pro
and finally we arrive at a formula that does not involve the arbitrary chosen object

Similarly we can prove a WFF $\forall x Q(x)$. We say for all object b, $Q(b)$ is true.
the application of rule C to a WFF $\exists x P(x)$ yields $P(c)$, then the object referred
by C may depend on the values of the free variables in $\exists x P(x)$. So that one obje
may not satisfy $P(x)$ for all values of the free variables in $\exists x P(x)$.

For example, we consider $\exists x P(x) \to Q(x)$, $\forall x P(x) \mapsto \exists x Q(x)$. To prove it, let
follow the steps given below :

Step 1 : $\exists x P(x) \to Q(x)$ **Step 2 :** $\forall x P(x)$ **Step 3 :** $P(c) \to Q(c)$ for some c

Step 4 : $P(c)$ **Step 5 :** $Q(c)$ **Step 6 :** $\exists x Q(x)$

Note that we can also achieve the same result without making an arbitrar
choice of an object c in step 3.

Example 3.35 : Let P be a first order formula. Then,

(a) $\sim(\exists x P(x)) \equiv \forall x(\sim P(x))$ (b) $\sim(\exists y P(y)) \equiv \forall y(\sim P(y))$

Illustrative Examples

1. **Prove the validity of the implication**
$\forall x (P(x) \to Q(x), \forall x(R(x) \to \sim Q(x) \Rightarrow \forall x(R(x) \to \sim P(x))$

Solution (i) $\forall x (P(x) \to Q(x))$ Rule P (given premise)
(ii) $(P(a) \to Q(a))$ Rule P ($\forall$ value $x = a$ in (i))
(iii) $\forall x (R(x) \to Q(x)$ Rule P (given premise)
(iv) $R(a) \to \sim Q(a)$ Rule P ($\forall$ value $x = a$ in (iii))
(v) $Q(a) \to \sim R(a)$ Rule T (Equivalence of (iv))
(vi) $P(a) \to \sim R(a)$ Rule T (Hypothetical Syllogism of (ii), (v))
(vii) $R(a) \to \sim P(a)$ Rule T (Equivalence of (vi))
(viii) $\forall x (R(x) \to \sim P(x))$ Rule T ($\forall$ value x in (vii))
Hence, the result follows.

2. **Show that the conclusion $\forall$**
$\exists x (P(x) \wedge Q(x)) \to \forall y (R(y)$

Solution (i) $\forall y (R(y) \wedge \sim S(y))$
(ii) $R(a) \wedge \sim S(a)$
(iii) $\sim R(a) \vee S(a)$
(iv) $\sim(R(a) \to S(a))$
(v) $\exists y(\sim R(y) \to S(y)$
(vi) $\sim\forall y(R(y) \to S(y)$
(vii) $\exists x (P(x) \wedge Q(x)) \to \forall y (R($
(viii) $\sim\exists x (P(x) \wedge Q(x))$
(ix) $\forall x \sim((P(x) \wedge Q(x))$
(x) $\sim (P(a) \wedge Q(a))$
(xi) $\sim P(a) \vee \sim Q(a)$
(xii) $P(a) \to \sim Q(a)$
(xiii) $\forall x (P(x) \to \sim Q(x))$
Hence, the result follows.

3. **Prove that**
$\forall x (P(x) \to (Q(y) \wedge R(x$

Solution (i) $\forall x (P(x) \to (Q(y$
(ii) $P(a) \to (Q(y) \wedge R(a))$
(iii) $\exists x P(x)$
(iv) $P(a)$
(v) $Q(y) \wedge R(a)$
(vi) $Q(y)$
(vii) $R(a)$
(viii) $P(a) \wedge R(a)$
(ix) $\exists x(P(x) \wedge R(x))$
(x) $Q(y) \wedge \exists x (P(x) \wedge R(x)$
Hence, the result follows.

4. **Let $P(x)$ denotes th**
integers, what is the

Solution (i) We have that
exists a value of x in the do
Thus, the existential quan
(ii) $\forall x P(x)$ is not true as fo
true

2. Show that the conclusion $\forall x(P(x) \to \neg Q(x))$ follows from the premises $\exists x\,(P(x) \wedge Q(x)) \to \forall y\,(R(y) \to S(y))$ and $\exists y\,(R(y) \wedge \neg S(y))$

Solution

(i)	$\forall y\,(R(y) \wedge \neg S(y))$	(given premise)
(ii)	$R(a) \wedge \neg S(a)$	($\exists$ value $x = a$ in (i))
(iii)	$\neg R(a) \vee S(a)$	(De Morgan's law of (ii))
(iv)	$\neg(R(a) \to S(a))$	(Equivalence of (iii))
(v)	$\exists y(\neg R(y) \to S(y)$	($\exists$ value $y = a$ in (iii))
(vi)	$\neg \forall y(R(y) \to S(y)$	(Negation equivalence of (v))
(vii)	$\exists x\,(P(x) \wedge Q(x)) \to \forall y\,(R(y) \to S(y))$	(given premise)
(viii)	$\neg \exists x\,(P(x) \wedge Q(x))$	(Modus tollens from (vi) and (vii))
(ix)	$\forall x \neg((P(x) \wedge Q(x))$	(Negative equivalence of (viii))
(x)	$\neg(P(a) \wedge Q(a))$	($\forall$ value $x = a$ in (ix))
(xi)	$\neg P(a) \vee \neg Q(a)$	(De Morgan's law)
(xii)	$P(a) \to \neg Q(a)$	(equivalence of (xi))
(xiii)	$\forall x\,(P(x) \to \neg Q(x))$	($\forall$ value $x = a$ in (xi))

Hence, the result follows.

3. Prove that
$\forall x\,(P(x) \to (Q(y) \wedge R(x))) \to S(y), \exists x\,(P(x) \Rightarrow Q(y) \wedge \exists x\,(P(x) \wedge R(x))$

Solution

(i)	$\forall x\,(P(x) \to (Q(y) \wedge R(x)))$	(given premise)
(ii)	$P(a) \to (Q(y) \wedge R(a))$	($\forall$ value $x = a$ in (i))
(iii)	$\exists x\,P(x)$	(given premise)
(iv)	$P(a)$	($\exists$ value $x = a$ in (iii))
(v)	$Q(y) \wedge R(a)$	(Modus pones (ii) and (iv))
(vi)	$Q(y)$	(Simplification (v))
(vii)	$R(a)$	(Simplification (v))
(viii)	$P(a) \wedge R\,(a)$	(Conjunction of (iv) and (vii))
(ix)	$\exists x(P(x) \wedge R\,(x))$	($\exists$ value $x = a$ in (viii))
(x)	$Q(y) \wedge \exists x\,(P(x) \wedge R(x))$	(Conjunction of (vi) and (ix))

Hence, the result follows.

4. Let $P(x)$ denotes the statement "$x = x^2$". If the domain consists of the integers, what is the truth value of (i) $\exists x\,P(x)$ (ii) $\forall x P(x)$. [VU, 2018]

Solution (i) We have that $x = 0$ and 1 satisfy the relation $x = x^2$. Hence, there exists a value of x in the domain for which $x = x^2$ holds.

Thus, the existential quantification of $P(x)$ that is $\exists x$, $P(x)$ is true.

(ii) $\forall x\,P(x)$ is not true as for all values of x in the domain the relation $x = x^2$ is not true

5. Let D = {1, 2, 3, ..., 9}. Determine the truth value of the following statements [W., 2011]
(i) $(\forall x \in D) x + 4 < 15$ (ii) $(\forall x \in D) x + 4 > 15$ (iii) $(\forall x \in D) x + 4 \leq 10$
(iv) $(\exists x \in D) x + 4 > 15$ (v) $(\exists x \in D) x + 4 = 10$

Solution (i) For all $x = 1, 2, 3, ..., 9$, we have $x + 4 < 15$, so $(\forall x \in D) x + 4 < 15$ is true

(ii) For all $x = 1, 2, 3, ..., 9$, we have $x + 4 > 15$ is not true for any x, so $(\forall x \in D) x + 4 > 15$ does not hold true.

(iii) Also $(\forall x \in D)\ x + 4 \leq 10$ is not true for $x = 7, 8, 9$.
So $(\forall x \in D)\ x + 4 \leq 10$ does not hold true $x \in D$.

(iv) $(\exists x \in D)\ x + 4 > 15$ is not true for for any x in D = {1, 2, 3, ..., 9}.

(v) $(\exists x \in D)\ x + 4 = 10$ is true for $x = 6$, so there exists at least one x for which $x + 4 = 10$ is true.

6. **Express in symbolic form using quantifiers $\lim_{x \to \infty} x_n = l$, where $\{x_n\}$ is a real sequence and l is a real number** [BU, 2011]

Solution A sequence $\{x_n\}$ converges to a real number l if for every $\varepsilon > 0$, there exists N such that $|x_n - l| < \varepsilon$, $\forall n \geq N$. The logical expression of this statement using quantifiers is

$\forall \varepsilon\, \exists\, N(\forall n \geq N \to |x_n - l| < \varepsilon)$, where the Universe of Discourse for ε, n consists of all positive real numbers and that for x consists of all real numbers.

7. **Find the truth value of $\forall x, P(x)$ where $P(x)$ is the statement "$x^2 < 50$" and the domain is the set {1, 2, 3, 4, 5}**

Solution We have, $1^2 = 1 < 50$, $2^2 = 4 < 50$, $3^2 = 9 < 50$, $4^2 = 16 < 50$ and $5^2 = 25 < 50$

So, $x^2 < 50$ hold for all values of x in the set {1, 2, 3, 4, 5}. Hence, the Universal quantifier $\forall x, P(x)$ is true.

8. **Find the truth value of $\forall x, P(x)$ where $P(x)$ is the statement "$2x + 1 < 10$" and the domain is the set {0, 2, 3, 5}**

Solution We have for $x = 0$, $2(0) + 1 = 1 < 10$,
$x = 2$, $2(2) + 1 = 5 < 10$,
$x = 3$, $2(3) + 1 = 7 < 10$,
$x = 5$, $2(5) + 1 = 11 > 10$
So, we cannot say $2x + 1 < 10$ for all values of x in {0, 2, 3, 5}
Hence, the Universal quantifier $\forall x, P(x)$ is not true.

9. **Find the truth value of the e... $P(x)$ is the statement "$x^2 > 3$... 6, 7, 8}**

Solution We have that $x = 0, 1, 2, 3$... exists a value of x in the domain for ... Thus, the existential quantificatio...

10. **Find the truth value of th... function $P(x, y)$ stating "x^2 ...**

Solution The statement $P(x, y)$: "... the variables take values from the d... $1^2 + 1^2 = 2 < 20$, $1^2 + 2^2 = 5 < 20$, 1^2... So $P(x, y)$: "$x^2 + y^2 < 20$" hold for ... Hence, the Universal quantifier ...

11. **Let $P(x, y)$ states that "$x^2 >$ ... Find the truth value of the**

Solution For $y = 1, 2, 3$, $y + 2 = 3$... Now, only for $x = 3$ in the dom... which is $x^2 = 3^2 = 9 > 3, 4, 5$.
Hence, there exists one value of ... So, $\exists x$ such that $\forall y\, P(x, y)$ is tru... Hence, $\exists x, \forall y\, P(x, y)$ is true.

12. **Let $P(x, y)$ states that "$x^2 +$ ... Find the truth value of th...**

Solution For $x = 1$, $1^2 + y^2 < 15$ ...
For $x = 2$, $2^2 + y^2 < 15$ or, y^2 ...
For $x = 3$, $3^2 + y^2 < 15$ or, y^2 ...
We see that for $y = 1$, $1^2 < 14$, 1...
Thus, there exists a value of y f...
Hence, $\forall x, \exists y\, P(x, y)$ is true.

13. **Determine the truth val... statement "$x^2 - 2x + 7 =$ ... discourse.**

Solution Here $P(x)$ is the state...
$x^2 - 2x + 7 = 0$
or, $x = \frac{2 \pm \sqrt{4 - 4 \times 1 \times 7}}{2}$ or, $x =$ ...

9. Find the truth value of the existential quantification of $\exists x$, $P(x)$, where $P(x)$ is the statement "$x^2 > 30$" and the domain is the set {0, 1, 2, 3, 4, 5, 6, 7, 8}

Solution We have that $x = 0, 1, 2, 3, 4, 5$ satisfy the relation $x^2 > 30$. Hence, there exists a value of x in the domain for which $x^2 > 30$ hold.
Thus, the existential quantification of $P(x)$ that is $\exists x$, $P(x)$ is true.

10. Find the truth value of the Universal quantifier of the propositional function $P(x, y)$ stating "$x^2 + y^2 > 10$" and the domain is the set {1, 2,3}

Solution The statement $P(x, y)$: "$x^2 + y^2 < 20$" contains two variables x, y. When the variables take values from the domain {1, 2, 3}, we have
$1^2+1^2=2<20$, $1^2 + 2^2 = 5 < 20$, $1^2 + 3^2 = 10 < 20$, $2^2 + 2^2 = 8 < 20$, $3^2 + 3^2 = 18 < 20$
So, $P(x, y)$: "$x^2 + y^2 < 20$" hold for all values of x in the set {1, 2, 3}.
Hence, the Universal quantifier $\forall x$, $P(x)$ is true.

11. Let $P(x, y)$ states that "$x^2 > y + 2$" where the domain is the set $S = \{1, 2, 3\}$. Find the truth value of the quantifier $\exists x, \forall y\, P(x, y)$.

Solution For $y = 1, 2, 3$, $y + 2 = 3, 4, 5$
Now, only for $x = 3$ in the domain $S = \{1, 2, 3\}$, the relation "$x^2 > y + 2$" hold, which is $x^2 = 3^2 = 9 > 3, 4, 5$.
Hence, there exists one value of x ($x = 3$) for which $x^2 > y + 2$ hold $\forall y$.
So, $\exists x$, such that $\forall y\, P(x, y)$ is true
Hence, $\exists x, \forall y\, P(x, y)$ is true.

12. Let $P(x, y)$ states that "$x^2 + y^2 < 15$" where the domain is the set $S = \{1, 2, 3\}$. Find the truth value of the quantifier $\forall x, \exists y\, P(x, y)$.

Solution For $x = 1$, $1^2 + y^2 < 15$ or, $y^2 < 14$
For $x = 2$, $2^2 + y^2 < 15$ or, $y^2 < 11$
For $x = 3$, $3^2 + y^2 < 15$ or, $y^2 < 6$
We see that for $y = 1$, $1^2 < 14$, $1^2 < 11$, $1^2 < 6$
Thus, there exists a value of y for which $x^2 + y^2 < 15$ hold for all values of x.
Hence, $\forall x, \exists y\, P(x, y)$ is true.

13. Determine the truth value of the quantifier $\exists x$, $P(x)$, where $P(x)$ is the statement "$x^2 - 2x + 7 = 0$" and set of real numbers is the Universe of discourse.

Solution Here $P(x)$ is the statement
$x^2 - 2x + 7 = 0$
or, $x = \frac{2 \pm \sqrt{4 - 4 \times 1 \times 7}}{2}$ or, $x = \frac{2 \pm \sqrt{-24}}{2}$ or, $x = 1 \pm i\sqrt{6}$

The two values of $x = 1 + i\sqrt{6}$ and $x = 1 - i\sqrt{6}$ are complex numbers.
Thus, there is no real value of x which satisfies the equation $x^2 - 2x + 7 = 0$
Hence, the existential quantifier $\exists x$, $P(x)$ is false.

14. **Translate the following statement into logical expression using predicates, quantifier and logical connectives :**
"At least one of your friends is perfect"

Solution Let $P(x)$ be the statement "your friend x is perfect", where the Universe of discourse consists of all friends. Then the mathematical statement is $\exists x P(x)$

15. **If $P(x, y)$ symbolizes the statement "x likes y", where the Universe of discourse for both x and y consists of all people in the world. Translate the following English sentences into logical expressions :**
(i) Everybody likes z.
(ii) Everybody likes somebody.
(iii) There is somebody whom everybody likes.
(iv) Nobody likes everybody.
(v) There is somebody whom no one likes.

Solution (i) $P(x, y)$ for all x. Hence, $\forall x\, P(x, z)$

(ii) $P(x, y)$ is true for all x and some y. Hence, $\forall x, \exists y\, P(x, y)$.

(iii) Even if, (iii) is the same as (ii), the stress is on the existence of somebody whom all "x" like. Hence, $\exists y, \forall x\, P(x, y)$.

(iv) Nobody likes everybody that is there is not one who likes everybody.
Hence, $\sim\exists x, \forall y\, P(x, y)$
$\equiv \forall x \sim\forall y\, P(x, y)$
$\equiv \forall x \exists y \sim P(x, y)$

(v) The sentence means that there is somebody whom everyone does not like.
Hence, $\sim\forall x, \exists y\, P(x, y)$
$\equiv \exists x \sim\exists y\, P(x, y)$
$\equiv \exists x \forall y \sim P(x, y)$

16. **Translate each statement using universal or existential quantifiers.**
"Every real number is either rational or irrational".

Solution Let us consider two statements $R(x)$ and $I(x)$ as follows :
$R(x)$: "x is rational number"
$I(x)$: "x is an irrational number"
The statement may be translated as $\forall x(R(x) \vee I(x))$.

17. **Express each of the following logical operations, predicates discourse consists of all stud honours course.**
(i) Every computer science
(ii) There is a student in the
(iii) Every student in the course.
(iv) There is a student mathematics course.

Solution (i) Let $P(x)$ be the state the Universe of discourse consist Then the mathematical statement
(ii) Let $P(x)$ be the statement "x of discourse consists of all statement is $\exists x\, P(x)$.
(iii) Let $P(x, y)$ be the statement of all students in the class a Then the mathematical stat
(iv) Let $P(x, y)$ be the statement of all students in the class a Then the mathematical sta

18. **Find the negation of th $(\forall x \in D)(x + 3 < 9)$ whe**

Solution $(\forall x \in D)(x + 3 < 9)$
$\equiv \sim(\forall x \in D)(x + 3 < 9)$
$\equiv (\exists x \in D) \sim(x + 3 < 9)$
$\equiv (\exists x \in D)(x + 3 \geq 9)$ whe

19. **Find a counter examp $(\forall x \in D)(x$ is prime) v**

Solution Here, for $x = 1$, the example.

20. **Express the negation in English :**
(i) If the teacher is
(ii) All the students
(iii) Some of the stu
(iv) No one has don

17. **Express each of the following statements using mathematical and logical operations, predicates and quantifiers, where the universe of discourse consists of all students of computer science / mathematics honours course.**

(i) Every computer science student needs a course in mathematics.

(ii) There is a student in the class who owns a personal computer.

(iii) Every student in the class has taken at least one mathematics course.

(iv) There is a student in the class who has taken at least one mathematics course.

Solution (i) Let $P(x)$ be the statement "x needs a course in mathematics", where the Universe of discourse consists of all computer science honours students. Then the mathematical statement is $\forall x\, P(x)$.

(ii) Let $P(x)$ be the statement "x owns a personal computer", where the Universe of discourse consists of all students in the class. Then the mathematical statement is $\exists x\, P(x)$.

(iii) Let $P(x, y)$ be the statement "x has taken y", where the Universe of x consists of all students in the class and that of y consists of all mathematics courses. Then the mathematical statement is $\forall x \exists y\, P(x, y)$.

(iv) Let $P(x, y)$ be the statement "x has taken y", where the Universe of x consists of all students in the class and that of y consists of all mathematical courses. Then the mathematical statement is $\exists x \exists y\, P(x, y)$.

18. **Find the negation of the following predicate :** [CU, 2021]
$(\forall x \in D)(x + 3 < 9)$ where $D = \{1, 2, 3, 4\}$

Solution $(\forall x \in D)(x + 3 < 9)$

$\equiv \sim(\forall x \in D)(x + 3 < 9)$

$\equiv (\exists x \in D) \sim(x + 3 < 9)$

$\equiv (\exists x \in D)(x + 3 \geq 9)$ where $D = \{1, 2, 3, 4\}$

19. **Find a counter example (if exist) for the statement :** [CU, 2021]
$(\forall x \in D)(x$ is prime) where $D = \{1, 3, 4, 7, 11\}$

Solution Here, for $x = 1$, the statement '1 is prime' is false. So, here 1 is a counter example.

20. **Express the negations of the following statements using quantifiers and in English :**

(i) If the teacher is absent, then some students do not keep quiet.

(ii) All the students keep quiet and the teacher is present.

(iii) Some of the students do not keep quiet or the teacher is absent.

(iv) No one has done every problem in the exercise.

Solution (i) Let P represents the teacher in the class and $Q(x)$ represents "x keeps quiet". Then the given statement is

$$\sim P \to \exists x \sim Q(x) \equiv \sim P \to \sim \forall x\, Q(x)$$
$$\equiv P \vee \sim \forall x\, Q(x)$$

Hence, Negation of the given statement is

$$\sim (P \vee \sim \forall x\, Q(x))$$
$$\equiv \sim P \wedge \forall x\, Q(x)$$

That is the teacher is absent and all the students keep quiet.

(ii) Let P represent the teacher in the class and $Q(x)$ represents "x keeps quiet". Then the given statement is

$$\forall x\, Q(x) \wedge P$$

Hence, Negation of the given statement is

$$\sim (\forall x\, Q(x) \wedge P)$$
$$\equiv \sim \forall x\, Q(x) \vee \sim P$$
$$\equiv \exists x \sim Q(x) \vee \sim P$$

That is some students do not keep quiet or the teacher is absent.

(iii) Let P represent the teacher in the class and $Q(x)$ represents "x keeps quiet". Then the given statement is

$$\exists x \sim Q(x) \vee \sim P$$
$$\equiv \sim \forall x\, Q(x) \vee \sim P$$

Hence, Negation of the given statement is

$$\sim (\sim \forall x\, Q(x) \vee \sim P)$$
$$\equiv \forall x\, Q(x) \wedge P$$

That is all the students keep quiet and the teacher is present.

(iv) Let $D(x, y)$ represent "x has done problem y".

Then the given statement is

$$(\sim \exists x)\ (\forall y\, D(x, y))$$

Hence, Negation of the given statement is

$$(\sim 172 \sim \exists(x))\ (\forall y\, D(x, y))$$
$$\equiv \exists(x)\ \forall y\, D(x, y)$$

That is someone has done every problem in the exercise.

21. **Determine the negation of the following statements**
(i) $\forall x \forall y \forall z, P(x, y, z)$ (ii) $\forall x \exists y, P(x, y)$ (iii) $\exists y \forall x \forall z, P(x, y, z)$

Solution (i) $\sim (\forall x \forall y \forall z, P(x, y, z))$
$\equiv \exists x \exists y \exists z \sim P(x, y, z)$

(ii) $\sim (\forall x \exists y, P(x, y))$
$\equiv \exists x \forall y, \sim P(x, y)$

(iii) $\sim (\exists y \forall x \forall z, P(x, y, z))$
$\equiv \forall y \exists x \exists z, \sim P(x, y, z)$

22. **Determine t**
(i) $\exists x \forall y$, (P(
(iii) $\forall x \forall y$, (

Solution (i) $\sim (\exists x$
$\equiv \forall x$
$\equiv \forall x$

(ii) $\sim (\forall x \exists y, (P(x, y$
$\equiv \exists x \forall y, \sim (P(x,$

(iii) $\sim (\forall x \forall y, (P(x)$
$\equiv \exists x \exists y, \sim (P(x$
$\equiv \exists x \exists y, (\sim P(x$

23. **Prove the**
$\forall x\, (P(x) \vee$

Solution This st
We shall prove t
Now, the follow

(i)	$\sim (\forall x\, (P($
(ii)	$\sim (\forall x\, (P($
(iii)	$\sim (\forall x\, (P($
(iv)	$\sim (\exists x\, Q($
(v)	$\exists x \sim (P($
(vi)	$\forall x \sim Q(x$
(vii)	$\sim P(a)$
(viii)	$\sim Q(a)$
(ix)	$\sim P(a) \wedge$
(x)	$\sim (P(a) \vee$
(xi)	$\forall x\, (P(x)$
(xii)	$P(a) \vee Q$
(xiii)	$(P(a) \vee$
(xiv)	F

This shows th
false. This contra
$\forall x\, (P(x) \vee$

24. **Prove th**
$\forall x\, (P\ ($

Solution This
We shall prov

22. **Determine the negation of the following statements**
(i) $\exists x \forall y, (P(x) \vee Q(y))$ **(ii)** $\forall x \exists y, (P(x, y) \rightarrow Q(x, y))$
(iii) $\forall x \forall y, (P(x) \wedge Q(y))$

Solution (i) $\sim (\exists x \forall y, (P(x) \vee Q(y))$
$\equiv \forall x \exists y, \sim(P(x) \vee Q(y))$
$\equiv \forall x \exists y, (\sim P(x) \wedge \sim Q(y))$

(ii) $\sim (\forall x \exists y, (P(x, y) \rightarrow Q(x, y))$
$\equiv \exists x \forall y, \sim(P(x, y) \rightarrow Q(x, y))$

(iii) $\sim(\forall x \forall y, (P(x) \wedge Q(y))$
$\equiv \exists x \exists y, \sim(P(x) \wedge Q(y))$
$\equiv \exists x \exists y, (\sim P(x) \vee \sim Q(y))$

23. **Prove the validity of the following argument using indirect method**
$\forall x (P(x) \vee Q(x)) \Rightarrow (\forall x (P(x)) \vee (\exists x Q(x))$

Solution This statement may be proved by the method of contradiction
We shall prove that $\sim (\forall x (P(x)) \vee (\exists x Q(x))$ is false.
Now, the following sequence of elementary arguments is used to give the result :

(i)	$\sim (\forall x (P(x)) \vee (\exists x Q(x))$	(given, assumed)
(ii)	$\sim (\forall x (P(x)) \wedge \sim ((\exists x Q(x))$	(De Morgan's law)
(iii)	$\sim (\forall x (P(x))$	(Simplification)
(iv)	$\sim (\exists x Q(x))$	(Simplification)
(v)	$\exists x \sim (P(x)$	(negation of (iii))
(vi)	$\forall x \sim Q(x)$	(negation of (iv))
(vii)	$\sim P(a)$	($\exists$ one value $x = a$ in (v))
(viii)	$\sim Q(a)$	($\forall$ value $x = a$ in (vi))
(ix)	$\sim P(a) \wedge \sim Q(a)$	(Conjunction of (vii)and (viii))
(x)	$\sim (P(a) \vee Q(a))$	(De Morgan's law)
(xi)	$\forall x (P(x) \vee Q(x))$	(given premise)
(xii)	$P(a) \vee Q(a)$	($\forall$ value $x = a$ in (xi))
(xiii)	$(P(a) \vee Q(a)) \wedge \sim (P(a) \vee Q(a))$	(conjunction of (x) and (xii))
(xiv)	F	(Complement law)

This shows that our assumption that $\sim (\forall x (P(x)) \vee (\exists x Q(x))$ is true is actually false. This contradiction proves that

$\forall x (P(x) \vee Q(x)) \Rightarrow (\forall x (P(x)) \vee (\exists x Q(x))$

24. **Prove the validity of the following argument using indirect method**
$\forall x (P(x) \rightarrow Q(x)) \wedge \exists y P(y) \Rightarrow \exists z Q(z)$

Solution This statement may be proved by the method of contradiction
We shall prove that $\sim(\exists z Q(z))$ is false

Now, the following sequence of elementary arguments is used to give the re

(i)	$\sim(\exists z\, Q(z))$	given, assumed
(ii)	$\forall z(\sim Q(z))$	Negation of (i)
(iii)	$(\sim Q(a))$	$\forall$ value $Z = a$ in (ii)
(iv)	$\exists y P(y)$	given premise
(v)	$P(a)$	$\exists$ one value $y = a$ in (iv)
(vi)	$P(a) \wedge \sim Q(a)$	conjunction of (iii) and (v)
(vii)	$\sim(\sim P(a) \vee Q(a))$	(De Morgan's law)
(viii)	$\sim(P(a) \to Q(a))$	(Equivalence of (vii))
(ix)	$\forall x\,(P(x) \to Q(x))$	(given premise)
(x)	$P(a) \to Q(a)$	$\forall$ value $x = a$ in (ix)
(xi)	$(P(a) \to Q(a)) \wedge \sim(P(a) \to Q(a))$	Conjunction of (viii) and (x)
(xii)	F	Complement law

This shows that our assumption that $\sim(\exists z\, Q(z))$ is true is actually false. Th contradiction proves that

$$\forall x\,(P(x) \to Q(x)) \wedge \exists y P(y) \Rightarrow \exists z\, Q(z)$$

25. **Show that $(\exists x)P(x)$ follows logically from the premises $(\forall x)(P(x) \to Q(x))$ and $(\exists x)P(x)$.** [VU, 20

Solution The steps for the derivation are as follows :

(1)	$(\exists x)\, P(x)$	Rule P (given premise)
(2)	$P(y)$	Rule ES, (1)
(3)	$(\forall x)(P(x) \to Q(x))$	Rule P (given premise)
(4)	$P(y) \to Q(y)$	Rule US, (3)
(5)	$Q(y)$	Rule T, (2), (4) and Modus Ponens
(6)	$(\exists x)Q(x)$	Rule EG, (5)

Hence, $(\exists x)Q(x)$ is a valid conclusion from the given premises.

26. **Prove that $(\forall x)(H(x) \vee R(x)) = (\forall x)H(x) \vee (\exists x)R(x)$**

Solution We introduce an additional (assumed) premise $\sim((\forall x)H(x) \vee (\exists x)R(x))$ and use the indirect method of proof.

(1)	$\sim((\forall x)H(x) \vee (\exists x)R(x))$	Rule P(assumed)
(2)	$\sim((\forall x)H(x) \wedge \sim(\exists x)R(x))$	Rule T, (1) and De Morgan's Law
(3)	$\sim((\forall x)H(x))$	Rule T, (2) and the tautology $(p \wedge q) \to p$
(4)	$(\exists x)\sim H(x)$	Rule T, (3) and the equivalence $\sim((\forall x)P(x) \equiv (\exists x)\sim P(x))$
(5)	$\sim(\exists x)R(x))$	Rule T, (2) and the tautology $(p \wedge q) \to p$
(6)	$(\forall x)\sim R(x)$	Rule T, (5) and the equivalence $\sim((\exists x)P(x)) \equiv (\forall x)\sim P(x)$
(7)	$\sim H(y)$	Rule ES, (4)

(8)	$\sim R(y)$	Rule US
(9)	$\sim H(y) \wedge \sim R(y)$	Rule T,
(10)	$\sim(H(y) \vee R(y))$	Rule T,
(11)	$(\forall x)(H(x) \vee R(x))$	Rule P
(12)	$H(y) \vee R(y)$	Rule
(13)	$\sim(H(y) \vee R(y)) \wedge (H(y) \vee R(y))$	Rule

Hence, we arrive at a contradictio

$\sim((\forall x)\, H(x) \vee (\exists x)R(x))$ is true

$\therefore ((\forall x)H(x) \vee R(x)) \equiv (\forall x)H(x) \vee$

27. **Suppose $p(x)$ and $q(x)$ are that $\forall x(p(x) \wedge q(x))$ and $\forall$**

Solution Left to the reader (sim

28. **Show $\sim P(r, s)$ is a valid $(\forall x)(\forall y)(P(x, y) \to Q(x, y$**

Solution We use the following as follows :

(1)	$(\forall x)(\forall y)(P(x, y) \to$
(2)	$(\forall y)P(r, y) \to Q(r, y)$
(3)	$P(r, s) \to Q(r, s)$
(4)	$\sim Q(r, s)$
(5)	$\sim P(r, s)$

Hence $\sim P(r, s)$ is a valid con

29. **Show that the followi**

$(\exists x)(P(x) \wedge Q(x)) \sim$

$(\exists y)(R(y) \to \sim S(y))$

$\therefore (\forall x)(P(x) \to \sim Q(x$

Solution The Steps for deriv as follows :

(1)	$(\exists y)(R(y) \wedge \sim S(y))$
(2)	$R(z) \wedge \sim S(z)$
(3)	$\sim(R(z) \to S(z))$
(4)	$(\exists y) \sim(R(y) \to S(y))$
(5)	$\sim(\forall y)(R(y) \to S(y))$

(9)	$\sim R(y)$	Rule US, (6)
(10)	$\sim H(y) \wedge \sim R(y)$	Rule T, (7), (8) and the tautology $((p)\wedge(q))\rightarrow(p \wedge q)$
(11)	$\sim(H(y) \vee R(y))$	Rule T, (9) and De Morgan's Law
(12)	$(\forall x)(H(x) \vee R(x))$	Rule P
(13)	$H(y) \vee R(y)$	Rule US,(11)
(14)	$\sim(H(y) \vee R(y)) \wedge (H(y) \vee R(y))$	Rule T, (10), (12) and the tautology $((p)\wedge(q)) \rightarrow (p \wedge q)$.

Hence, we arrive at a contradiction. Thus assumption that
$\sim((\forall x)\, H(x) \vee (\exists x)R(x))$ is true is actually false
$\therefore ((\forall x)H(x) \vee R(x)) \equiv (\forall x)H(x) \vee (\exists x)R(x)$.

27. **Suppose $p(x)$ and $q(x)$ are two predicates defined on a domain D. Show that $\forall x(p(x) \wedge q(x))$ and $\forall x p(x) \wedge \forall x q(x)$ are logically equivalent.**

[CU, 2021]

Solution Left to the reader (similar as problem 26)

28. **Show $\sim P(r, s)$ is a valid conclusion that follows from the premises $(\forall x)(\forall y)(P(x, y) \rightarrow Q(x, y))$ and $\sim Q(r, s)$.**

Solution We use the following Steps for deriving that $\sim P(r, s)$ is valid conclusion as follows :

(1)	$(\forall x)(\forall y)(P(x, y) \rightarrow Q(x, y))$	Rule P
(2)	$(\forall y)P(r, y) \rightarrow Q(r, y)$	Rule US, (1)
(3)	$P(r, s) \rightarrow Q(r, s)$	Rule US, (2)
(4)	$\sim Q(r, s)$	Rule P
(5)	$\sim P(r, s)$	Rule T, (3), (4) and the Modus Tollens

Hence $\sim P(r, s)$ is a valid conclusion from the given premises.

29. **Show that the following argument is valid :**

$$\frac{(\exists y)(R(y) \rightarrow \sim S(y))}{\therefore (\forall x)(P(x) \rightarrow \sim Q(x))}$$

Solution The Steps for deriving the given conclusion from the given premises are as follows :

(1)	$(\exists y)(R(y) \wedge \sim S(y))$	Rule P
(2)	$R(z) \wedge \sim S(z)$	Rule ES, (1)
(3)	$\sim(R(z) \rightarrow S(z))$	Rule T, (2) and the equivalence $\sim(p \rightarrow q) \equiv p \wedge \sim q$
(4)	$(\exists y) \sim(R(y) \rightarrow S(y))$	Rule EG, (3)
(5)	$\sim(\forall y)(R(y) \rightarrow S(y))$	Rule T, (4) and the equivalence $(\exists x) \sim A(x) \equiv \sim(\forall x)A(x)$

(6)	$(\exists x)(P(x) \wedge Q(x)) \rightarrow (\forall y)(R(y) \rightarrow S(y))$	Rule P
(7)	$\sim(\exists x)(P(x) \wedge Q(x))$	Rule T, (5), (6) and Modus Tollens
(8)	$(\forall x)\sim(P(x) \wedge Q(x))$	Rule T, (7) and the equivalence $\sim(\exists x)A(x) \equiv (\forall x)\sim A(x)$
(9)	$\sim(P(y) \wedge Q(y))$	Rule US, (8)
(10)	$P(y) \rightarrow \sim Q(y)$	Rule T, (9) and the equivalence $\sim(p \wedge q) \equiv \sim p \vee \sim q$ and $p \rightarrow q \equiv \sim p \vee q$
(11)	$(\forall x)(P(x) \rightarrow \sim Q(x))$	Rule UG, (10)

Thus $(\forall x)(P(x) \rightarrow \sim Q(x))$ is a valid conclusion from the given premises.
Hence the given argument is valid.

30. **Verify the validity of the following argument.**
"Every living thing is a plant or an animal. John's jelly fish is alive and it is not a plant. All animals have souls. Therefore John's jelly fish has a soul".

Solution Let us consider the following :
$P(x)$: x is a plant
$A(x)$: x is an animal
$H(x)$: x has a soul
g : John's jelly fish
Then the interference pattern is $(\forall x)(P(x) \vee A(x))$

$$\frac{\sim P(g)}{\frac{(\forall x)((A(x) \rightarrow H(x))}{H(g)}}$$

Argument

(1)	$(\forall x)(p(x) \vee A(x))$	Rule P
(2)	$\sim P(g)$	Rule P
(3)	$P(g) \vee A(g)$	Rule US, (1)
(4)	$A(g)$	Rule T, (2),(3)
(5)	$(\forall x)(A(x) \rightarrow H(x))$	Rule P
(6)	$A(g) \rightarrow H(g)$	Rule US,(5)
(7)	$H(g)$	Rule T,(4), (6)

Thus the conclusion is valid.

31. **Verify the validity of the following argument :**
"Tigers are dangerous animals. There are Tigers. Therefore, there are dangerous animals".

Solution Let $T(x)$: x is tiger
$D(x)$: x is dangerous animal
Then the interference pattern is $(\forall x)$
$(\exists x)$
$(\exists x)$

Argument

(1)	$(\exists x)(T(x))$	Rule
(2)	$T(b)$	Rule
(3)	$(\forall x)(T(x) \rightarrow D(x))$	Rule
(4)	$T(b) \rightarrow D(b)$	Rule
(5)	$D(b)$	Rule
(6)	$(\exists x)(D(x))$	Rule

Thus the inference is valid.

32. **Given an argument which**
inference :
"All integers are rational nu
Some integers are powers
Therefore, some rational n

Solution Let us consider the follo
Let $P(x)$: x is an integer
$R(x)$: x is a rational number
$S(x)$: x is power of 2
Then the given inference pattern

Argument

(1)	$(\exists x)(P(x) \wedge S(x))$
(2)	$P(b) \wedge S(b)$
(3)	$P(b)$
(4)	$S(b)$
(5)	$(\forall x)(P(x) \rightarrow R(x))$
(6)	$P(b) \rightarrow R(b)$
(7)	$R(b)$
(8)	$R(b) \wedge S(b)$
(9)	$(\exists x)(R(x) \wedge S(x))$

Thus the inference is valid.

Then the interference pattern is $(\forall x)(T(x) \rightarrow D(x))$

$$\frac{(\exists x)(T(x))}{(\exists x)(D(x))}$$

Argument

(1)	$(\exists x)(T(x))$	Rule P
(2)	$T(b)$	Rule ES, (1)
(3)	$(\forall x)(T(x) \rightarrow D(x))$	Rule P
(4)	$T(b) \rightarrow D(b)$	Rule US, (3)
(5)	$D(b)$	Rule T, (2), (4)
(6)	$(\exists x)(D(x))$	Rule EG, (5)

Thus the inference is valid.

32. **Given an argument which will establish the validity of the following inference :**

"All integers are rational numbers

Some integers are powers of 2

Therefore, some rational numbers are powers of 2"

Solution Let us consider the following :

Let $P(x)$: x is an integer

$R(x)$: x is a rational number

$S(x)$: x is power of 2

Then the given inference pattern is $(\forall x)(P(x) \rightarrow R(x))$

$$\frac{(\exists x)(P(y) \wedge S(x))}{\therefore (\exists x)(R(x) \wedge S(x))}$$

Argument

(1)	$(\exists x)(P(x) \wedge S(x))$	Rule P
(2)	$P(b) \wedge S(b)$	Rule ES, (1)
(3)	$P(b)$	Rule T, (2)
(4)	$S(b)$	Rule T, (2)
(5)	$(\forall x)(P(x) \rightarrow R(x))$	Rule P
(6)	$P(b) \rightarrow R(b)$	Rule US, (5)
(7)	$R(b)$	Rule T, (3), (6)
(8)	$R(b) \wedge S(b)$	Rule T, (7), (4)
(9)	$(\exists x)(R(x) \wedge S(x))$	Rule EG, (8)

Thus the inference is valid.

33. **Show that (a) $(\exists x)(F(x) \wedge S(x)) \rightarrow (\forall y)(M(y) \rightarrow W(y))$ (b) $(\exists y)(M(y) \wedge \neg W(y))$ The conclusion $(\forall x)(F(x) \rightarrow \neg S(x))$ follows**

Solution Let us consider the following :

(1)	$(\exists y)(M(y) \wedge \neg W(y))$	Rule P
(2)	$M(z) \wedge \neg W(z)$	Rule ES, (1)
(3)	$\neg(M(z) \rightarrow W(z))$	Rule T, (2)
(4)	$(\exists y)\neg(M(y) \rightarrow W(y))$	Rule EG, (3)
(5)	$\neg(\forall y)(M(y) \rightarrow W(y))$	(4)
(6)	$(\exists x)(F(x) \wedge S(x)) \rightarrow (\forall y)(M(y) \rightarrow W(y))$	Rule P
(7)	$(\exists x)(F(x) \wedge S(x))$	Rule T, (5), (6)
(8)	$(\forall x) \neg(F(x) \wedge S(x))$	Rule T, (7)
(9)	$\neg(F(x) \wedge S(x))$	Rule US, (8)
(10)	$F(x) \rightarrow \neg S(x)$	Rule T, (9)
(11)	$(\forall x)(F(x) \rightarrow \neg S(x))$	Rule UG, (10)

Hence, the result follows.

Exercise

A Short and Long Answer Type Questions

1. Express each of the following statements using mathematical and logical operations, predicates and quantifiers, where the universe of discourse consists of all students of a school.
   (i) There is a student in the school who can speak English and who knows Bengali
   (ii) No student in the school can speak English or knows Bengali
   (iii) There is a student in the school who can speak English but who does not know Bengali.
   (iv) All the students in the school can speak English as well as know Bengali
2. Let $P(x)$ be the statement "x can speak English", $Q(x)$ be the statement "x knows Bengali", where the universe of discourse consists of all students of a school. Translate the following English sentences into logical expressions.
   (i) $\forall x(P(x) \vee Q(x))$
   (ii) $\exists x(\neg P(x) \vee \neg Q(x))$
   (iii) $\forall x \neg(P(x) \wedge Q(x))$
   (iv) $\forall x(P(x) \vee \neg Q(x))$
3. Let $P(x)$ be the statement "x is a shark", $Q(x)$ be the statement "x is a fish" and $R(x)$ be "x lives in water", where the universe of discourse consists of all animals. Translate the following English sentences into logical expressions.
   (i) $\exists x (\neg R(x))$

(ii) $\exists x(Q(x) \wedge \neg P(x))$
(iii) $\forall x (P(x) \wedge R(x)) \rightarrow Q(x)$
(iv) $\forall x (P(x) \wedge Q(x)) \rightarrow R(x)$

4. Symbolize the following sentence
   i. All men are mortal
   ii. It is false that for all natural nu
   iii. Everybody is not rich.
   iv. Every natural number is either
   v. There is an integer which is b
   vi. For any integer, there exists a zero.
   vii. Every integer is divisible by 6
5. Determine the truth values of t
   (i) If $a - b = 0$, then $a^2 - b^2 = 0$
   (ii) 2 is even and 5 is even
   (ii) If $4 > 5$, then $7 > 6$
   (iii) $4 < 5$ and 4 is a positive inte
6. Prove the following implicatio
   (i) $\forall x (P(x) \rightarrow Q(x)) \wedge \forall x (Q(x$
   (ii) $\exists x P(x), \forall x (P(x) \rightarrow Q(x)) =$
   (iii) $\exists x(P(x) \wedge Q(x)) \Rightarrow \exists x(P(x)$
   (iv) $\exists x P(x) \rightarrow \forall x Q(x) \Rightarrow \forall x(P$
   (v) $\forall x (P(x) \rightarrow Q(x)) \Rightarrow \forall x (P($
   (vi) $\forall x (C(x) \rightarrow A(x)) \Rightarrow \forall x (\exists y$
7. Translate these statements consists of all real numbers.
   (i) $\forall x \exists y(x < y)$
   (ii) $\forall x \forall y (((x \geq 0) \wedge (y \geq 0)) -$
   (iii) $\forall x \forall y \exists z (xy = z)$
   (iv) $\exists x \forall y (xy = z)$
   (v) $\forall x \forall y (((x \geq 0) \wedge (y \geq 0)) -$
   (vi) $\forall x \forall y \exists z(x = y + z)$
8. What is Prenex Normal For
   $(\forall x (\exists y R(x, y) \wedge \forall y) \sim S(x,$
9. Show whether the followin
   (i) Zebras eat grass
   Tigers eat Zebras
   Tigers eat grass

(ii) $\exists x(Q(x) \wedge \neg P(x))$
(iii) $\forall x\,(P(x) \wedge R(x)) \rightarrow Q(x)$
(iv) $\forall x\,(P(x) \wedge Q(x)) \rightarrow R(x)$

4. Symbolize the following sentence using predicates and quantifiers :
   i. All men are mortal
   ii. It is false that for all natural numbers n, $n + 5 < 10$
   iii. Everybody is not rich.
   iv. Every natural number is either even or odd.
   v. There is an integer which is both even and prime.
   vi. For any integer, there exists an integer such that their product is equal to zero.
   vii. Every integer is divisible by 6 if and only if it is divisible both by 2 and 3.

5. Determine the truth values of the following statements [VU, 2020]
   (i) If $a - b = 0$, then $a^2 - b^2 = 0$
   (ii) 2 is even and 5 is even
   (ii) If $4 > 5$, then $7 > 6$
   (iii) $4 < 5$ and 4 is a positive integer

6. Prove the following implications
   (i) $\forall x\,(P(x) \rightarrow Q(x)) \wedge \forall x\,(Q(x) \rightarrow R(x)) \Rightarrow \forall x\,(P(x) \rightarrow R(x))$
   (ii) $\exists x\,P(x), \forall x\,(P(x) \rightarrow Q(x)) \Rightarrow \exists x\,Q(x)$
   (iii) $\exists x(P(x) \wedge Q(x)) \Rightarrow \exists x(P(x) \wedge \exists x\,Q(x))$
   (iv) $\exists x\,P(x) \rightarrow \forall x\,Q(x) \Rightarrow \forall x(P(x) \rightarrow Q(x))$
   (v) $\forall x\,(P(x) \rightarrow Q(x)) \Rightarrow \forall x\,(P(x) \rightarrow \forall x\,Q(x))$
   (vi) $\forall x\,(C(x) \rightarrow A(x)) \Rightarrow \forall x\,(\exists y(C(y) \wedge B(x, y)) \rightarrow \exists y\,(A(y) \wedge B(x, y)))$

7. Translate these statements into English, where domain of each variable consists of all real numbers.
   (i) $\forall x \exists y(x < y)$
   (ii) $\forall x \forall y\,(((x \geq 0) \wedge (y \geq 0)) \rightarrow (xy \geq 0))$
   (iii) $\forall x \forall y \exists z\,(xy = z)$
   (iv) $\exists x \forall y\,(xy = z)$
   (v) $\forall x \forall y\,(((x \geq 0) \wedge (y \geq 0)) \rightarrow (x - y > 0))$
   (vi) $\forall x \forall y \exists z(x = y + z)$

8. What is Prenex Normal Form. Find the Prenex Normal form of
   $(\forall x\,(\exists y\,R(x, y) \wedge \forall y) - S(x, y) \rightarrow \neg(\exists y\,R(x, y) \wedge P))$

9. Show whether the following arguments are valid :
   (i) Zebras eat grass
   Tigers eat Zebras

   Tigers eat grass

(ii) No human beings have tails
All herbivorous have tails
All herbivorous are vegetarian.
All human beings are not vegetarian.

(iii) All men are mortal
Socrates is a man.
Socrates is mortal

(iv) All divisor of 8 are also divisor of 24
7 is not a divisor of 24
7 is not a divisor of 8

(v) All even integers are divisible by 2
8 is an even integer.
8 is divisible by 2

(vi) Every differentiable function is continuous
ϕ is continuous.
ϕ is differentiable.

10. Test the logical validity of the given argument : All integers are rational numbers. All rational numbers are real numbers. 2 is an integer. Therefore, 2 is a real number [CU. 2021]

11. Remove as many as brackets as possible from the following statement formula $((\sim(P \vee R) \vee (\sim(Q \leftrightarrow P))))$ [CU. 2021]

12. Test the logical validity of each of the following arguments :

(i) All the divisors of 6 are also divisors of 12.
3 is a divisor od 6.
Therefore, 3 is a divisor of 12.

(ii) All integers are rational numbers
All rational numbers are real numbers
3 is an integer.
Therefore, 3 is a real number.

(iii) Some odd integers are prime.
9 is an odd integer.
Therefore, 9 is prime.

(iv) Some students work hard.
All females work hard.
Therefore, some students are females.

13. Verify the validity of the interferen

(i) If one person is more successful
to deserve success. Suvo has
Suvo is not more successful th

(ii) All integers are rational numb
some rational numbers are p

(iii) A student in this class has n
in this class passed the Ma
who passed the Mathematic
Logic book.

14. Translate each of the followi
predicates, quantifier and lo

(i) No Physics students know
(ii) All Mathematics students
(iii) Not every Physics studen
(iv) Atleat one Mathematics
(v) No Physics students no

15. Translate the following sta
$\forall x((x \neq 0) \rightarrow \exists y(xy = 1))$

16. Determine the truth valu
$A = \{1, 2, 3, 4, 5, 6\}$ is the
(i) $\exists x\,(x + 3 = 12)$
(ii) $\forall x\,(x + 3 < 12)$
(iii) $\exists x\,(x + 3 < 6)$
(iv) $\forall x\,(x + 3 \leq 8)$
(v) $\exists x\,(x^2 - 9 = 0)$

17. Determine the truth va
$A = \{1, 2, 3\}$ is the Univ
(i) $\exists x, \forall y, x^2 < y + 1$
(ii) $\forall x, \exists y, x^2 + y^2 < 12$

18. Determine the truth v
$A = \{1, 2, 3, \ldots, 9, 10\}$ i
(i) $\forall x, \exists y, (x + y < 14)$
(ii) $\forall x, \forall y, (x + y < 14$

19. Symbolize the follow
connectives:
A negative real num

13. Verify the validity of the interference.
(i) If one person is more successful than the other, then he has worked harder to deserve success. Suvo has not worked harder than Raja. Therefore, Suvo is not more successful than Raja.
(ii) All integers are rational numbers. Some integers are power of 2. Therefore, some rational numbers are power of 2.
(iii) A student in this class has not read Mathematical Logic book. Everyone in this class passed the Mathematical Logic exam. Therefore, someone who passed the Mathematical Logic exam has not read the Mathematical Logic book.

14. Translate each of the following expressions into logical expressions using predicates, quantifier and logical connectives:
(i) No Physics students know C++
(ii) All Mathematics students know C++
(iii) Not every Physics students know C++
(iv) Atleat one Mathematics student know C++
(v) No Physics students nor Mathematics student know C++ [VU, 2018]

15. Translate the following statement into simple English :
$\forall x((x \neq 0) \rightarrow 3y(xy = 1))$ [CU, 2021]

16. Determine the truth value of each of the following statements where $A = \{1, 2, 3, 4, 5, 6\}$ is the Universe of discourse.
(i) $\exists x\,(x + 3 = 12)$
(ii) $\forall x\,(x + 3 < 12)$
(iii) $\exists x\,(x + 3 < 6)$
(iv) $\forall x\,(x + 3 \leq 8)$
(v) $\exists x\,(x^2 - 9 = 0)$

17. Determine the truth value of each of the following statements where $A = \{1, 2, 3\}$ is the Universe of discourse.
(i) $\exists x, \forall y, x^2 < y + 1$
(ii) $\forall x, \exists y, x^2 + y^2 < 12$

18. Determine the truth value of each of the following statements where $A = \{1, 2, 3, ..., 9, 10\}$ is the Universe of discourse.
(i) $\forall x, \exists y, (x + y < 14)$
(ii) $\forall x, \forall y, (x + y < 14)$

19. Symbolize the following sentence using predicates, quantifiers and logical connectives:
A negative real number does not have a square root that is real number. [CU, 2021]

20. Determine the truth values of the statements if the domain of all variables consists of all integers
(i) $\forall n \exists m\ (n^2 < m)$
(ii) $\exists n \forall m\ (n < m^2)$
(iii) $\forall n \exists m\ (n + m = 0)$
(iv) $\exists n \exists m\ (n^2 + m^2 = 5)$
(v) $\exists n \exists m\ (n + m = 4, n - m = 1)$ [MU, 20

21. Let D = {1, 2, 3, 4, 5} be the domain of the variables x, y and z. Find the truth value of the following statements :
(i) $\exists x \exists y (x^2 > y + 3)$
(ii) $\forall x \exists y (x^2 + y^2 < 25)$
(iii) $\forall x \forall y \exists z (x^2 + y^2 > 4z^2)$
(iv) $\exists x \forall y \exists z (x^2 + y^2 + z^2 < 50)$

22. Show whether the following arguments are valid :
(i) What is good for the auto industry is good for the country. What is good for the country is good for you. What is good for the auto industry is good for you to buy an expensive car. Therefore, it is good for the country and for you to buy an expensive car.
(ii) It is not the case that some trigonometric functions are not periodic. Some periodic functions are continuous. Therefore, it is not the case that all trigonometric functions are not continuous.

23. Find the truth set of the following predicate
$(\forall y \in D)(x^2 + y < 12)$, where domain of x is the set of positive integers [CU, 2021

24. Symbolize the following :
(i) Some reals are not rational
(ii) A professor is a good teacher if he is both well informed and well behaved [BU, 2019

25. Determine the truth value of each of the following statements where the set of all numbers is the Universe of discourse
(i) $\forall x\, P(x)$ where $P(x)$ is "$x^2 = x^{n}$"
(ii) $\exists x\, P(x)$ where $P(x)$ is "$2x = x^{n}$"
(iii) $\exists x\, P(x)$ where $P(x)$ is "$x^2 - 4 = 0$"

26. Show that
(i) $(x)(P(x) \to Q(x)) \wedge (x)(Q(x) \to R(x)) \Rightarrow (x)(P(x) \to R(x))$
(ii) $(x)(P(x) \vee Q(x)) \Rightarrow (x)P(x) \vee (\exists x)Q(x)$
(iii) $(x)(H(x) \to M(x)) \wedge H(y) \Rightarrow M(y)$
(iv) $(\forall x)(P(x) \vee Q(x)), (\forall x) \sim P(x) \Rightarrow (\forall x)(Q(x))$
(v) $(\exists x)(P(x) \wedge Q(x)) \Rightarrow (\exists x)P(x) \wedge (\exists x)Q(x)$

27. Express the negation of the statement using "Every student in this class is intelligent"

28. Determine the negation of the following
(i) x is real number, if $x > 3$ then $x^2 > 9$
(ii) x is real number, if $x(x + 1) > 0$ then x

29. Find the negation of the following state
(i) $\exists x\, P(x) \wedge \exists y\, Q(y)$
(ii) $\forall x\, P(x) \vee \exists y\, Q(y)$
(iii) $\forall x \forall y \exists z\, P(x, y, z)$
(iv) $\exists x \forall y \exists z\, P(x, y, z)$
(v) $\forall x \exists y\, (P(x, y) \to Q(x, y)))$
(vi) $\exists x \exists y\, (P(x) \wedge \sim Q(y))$
(vii) $\exists y\, P(y) \to \forall x \sim Q(x)$

30. Determine the negation of the follow
(i) $\sim \forall x \forall y\, P(x, y)$
(ii) $\sim (\exists x \exists y \sim R(x, y) \wedge \forall x \forall y\, P(x, y)$

31. Is the following conclusion valid de
If $(\forall x)(P(x) \to Q(x)), (\exists y)P(y)$ then $(\exists$

32. State the completeness theorem of

33. Define free and bound variable in a bound variable(s) in the following
$(\forall x_1)A_1^2(x_1, x_2) \to (\forall x_1)A_1^1(x_1))$

34. Prove that $\to (\forall x)(B \leftrightarrow C) \to ((\forall x)$ Predicate logic.

B Multiple Choice Questions (MCQ)

1. **Indicate which of the following w**
(a) $((P \Rightarrow Q) \wedge (Q \Rightarrow R)) \Rightarrow (P \Rightarrow R$
(c) $(P \wedge (\sim P \vee \sim Q)) \Rightarrow Q$

2. **Which of the following predicate**
(a) $(\forall x)P(x) \vee (\forall x)Q(x) \to (\forall x)(P($
(b) $(\exists x)P(x) \wedge (\exists x)Q(x) \to (\exists x)(P(x$
(c) $(\exists x)(P(x) \vee Q(x)] \to (\forall x)P(x) \vee$
(d) $(\exists x)(P(x) \vee Q(x)] \to \exists x\, P(x) \vee$

3. **Consider two well-formed formu**
$F_1 : P \Rightarrow \sim P$
$F_2 : (P \Rightarrow \sim P) \vee (\sim P \Rightarrow P)$
Which of the following statemer
(a) F_1 is satisfiable, F_2 is valid

27. Express the negation of the statement using quantifiers and in English : "Every student in this class is intelligent" in two different ways

28. Determine the negation of the following statements
    (i) x is real number, if $x > 3$ then $x^2 > 9$
    (ii) x is real number, if $x(x + 1) > 0$ then $x > 0$ or $x < -1$

29. Find the negation of the following statements:
    (i) $\exists x\, P(x) \wedge \exists y\, Q(y)$
    (ii) $\forall x\, P(x) \vee \exists y\, Q(y)$
    (iii) $\forall x\, \forall y\, \exists z\, P(x, y, z)$
    (iv) $\exists x\, \forall y\, \exists z\, P(x,y,z)$
    (v) $\forall x\, \exists y\, (P(x,y) \rightarrow Q(x, y)))$
    (vi) $\exists x\, \exists y\, (P(x) \wedge \neg Q(y))$
    (vii) $\exists y\, P(y) \rightarrow \forall x \neg Q(x)$

30. Determine the negation of the following statements
    (i) $\neg \forall x\, \forall y\, P(x, y)$
    (ii) $\neg(\exists x\, \exists y \neg R(x, y) \wedge \forall x\, \forall y\, P(x, y)$

31. Is the following conclusion valid derivable from the premises given?
    If $(\forall x)(P(x) \rightarrow Q(x))$, $(\exists y)P(y)$ then $(\exists z)Q(z)$

32. State the completeness theorem of Predicate logic. [CU, 2021]

33. Define free and bound variable in a well formed formula. Find the free and bound variable(s) in the following well defined formula
    $(\forall x_1)A_1^2(x_1, x_2) \rightarrow (\forall x_1)A_1^1(x_1))$ [CU, 2021]

34. Prove that $\rightarrow (\forall x)(B \leftrightarrow C) \rightarrow ((\forall x)B \leftrightarrow (\forall x)C)$, where B, C are well formed in Predicate logic. [CU, 2021]

B Multiple Choice Questions (MCQ)

1. **Indicate which of the following well-formed formula are valid :**
    (a) $((P \Rightarrow Q) \wedge (Q \Rightarrow R)) \Rightarrow (P \Rightarrow R)$
    (b) $(P \Rightarrow Q) \Rightarrow (\neg P \Rightarrow \neg Q)$
    (c) $(P \wedge (\neg P \vee \neg Q)) \Rightarrow Q$
    (d) $((P \Rightarrow R) \vee (Q \Rightarrow R)) \Rightarrow ((P \vee Q) \Rightarrow R)$

2. **Which of the following predicate calculus statements is/ are valid :**
    (a) $(\forall x)P(x) \vee (\forall x)Q(x) \rightarrow (\forall x)[P(x) \vee Q(x)]$
    (b) $(\exists x)P(x) \wedge (\exists x)Q(x) \rightarrow (\exists x)[P(x) \wedge Q(x)]$
    (c) $(\exists x)[P(x) \vee Q(x)] \rightarrow (\forall x)P(x) \vee (\forall x)Q(x)$
    (d) $(\exists x)[P(x) \vee Q(x)] \rightarrow \exists x\, P(x) \vee \forall x\, Q(x)$

3. **Consider two well-formed formulas in propositional logic :**
    $F_1 : P \Rightarrow \neg P$
    $F_2 : (P \Rightarrow \neg P) \vee (\neg P \Rightarrow P)$
    Which of the following statements is correct?
    (a) F_1 is satisfiable, F_2 is valid

(b) F_1 is unsatisfiable, F_2 is satisfiable.
(c) F_1 is unsatisfiable, F_2 is valid
(d) F_1 and F_2 are both satisfiable.

4. Which of the following is valid first order formula? (Here α and β are ... order with x as their only free variable)

(a) $((\forall x)[\alpha] \Rightarrow (\forall x)[\beta]) \Rightarrow (\forall x)[\alpha \Rightarrow \beta]$ (b) $(\forall x)[\alpha] \Rightarrow (\exists x)[\alpha \wedge \beta]$
(c) $((\forall x)[\alpha \vee \beta] \Rightarrow (\exists x)[\alpha] \Rightarrow (\forall x)[\alpha]$ (d) $((\forall x)[\alpha \Rightarrow \beta] \Rightarrow (\forall x)[\alpha]) \Rightarrow [\forall x$

5. Identify the correct translation into logical notation of the following assertion.

"Some boys in the class are taller than all the girls."

Note : Taller (x, y) is true if x is taller than y.

(a) $(\exists x)(\text{boy}(x) \rightarrow (\forall y)(\text{girl}(y) \wedge \text{taller}(x, y)))$
(b) $(\exists x)(\text{boy}(x) \wedge (\forall y)(\text{girl}(y) \wedge \text{taller}(x, y)))$
(c) $(\exists x)(\text{boy}(x) \rightarrow (\forall y)(\text{girl}(y) \rightarrow \text{taller}(x, y)))$
(d) $(\exists x)(\text{boy}(x) \wedge (\forall y)(\text{girl}(y) \rightarrow \text{taller}(x, y)))$

6. Let $a(x, y)$, $b(x,y)$ and $c(x, y)$ be three statements with variables x and y chosen from some universe. Consider the following statement :

$$(\exists x)(\forall y)[(a(x, y) \wedge b(x, y)) \wedge \sim c(x, y)]$$

Which one of the following is its equivalent?

(a) $(\forall x)(\exists y)[(a(x,y) \vee b(x,y)) \rightarrow c(x, y)]$
(b) $(\exists x)(\forall y)[(a(x,y) \vee b(x,y)) \rightarrow c(x, y)]$
(c) $\sim(\forall x)(\exists y)[(a(x,y) \wedge b(x,y) \rightarrow c(x, y)]$
(d) $\sim(\forall x)(\exists y)[(a(x,y) \vee b(x,y) \rightarrow c(x, y)]$

7. The following propositional statement is $P \rightarrow (Q \vee R)) \rightarrow ((P \wedge Q) \rightarrow R)$

(a) Satisfy but not valid (b) Valid
(c) A contradiction (d) None of the above

8. What is the first order predicate calculus statement equivalent to the following? "Every teacher is liked by some student".

(a) $\forall(x)[\text{teacher}(x) \rightarrow \exists(y)[\text{student}(y) \rightarrow \text{likes}(y, x)]]$
(b) $\forall(x)[\text{teacher}(x) \rightarrow \exists(y)[\text{student}(y) \wedge \text{likes}(y, x)]]$
(c) $\exists(y)\forall(x)[\text{teacher}(x) \rightarrow [\text{student}(y) \wedge \text{likes}(y, x)]]$
(d) $\forall(x)[\text{teacher}(x) \wedge \exists(y)[\text{student}(y) \rightarrow \text{likes}(y, x)]]$

9. Let $P(x)$ and $Q(y)$ be arbitrary predicates. Which of the following statement is always TRUE?

(a) $(\forall x(P(x) \vee Q(x))) \Rightarrow (\forall x P(x)) \vee (\forall x Q(x)))$
(b) $(\forall x(P(x) \Rightarrow Q(x))) \Rightarrow (\forall x P(x)) \Rightarrow (\forall x Q(x)))$
(c) $(\forall x(P(x) \Rightarrow (\forall x Q(x))) \Rightarrow (\forall x P(x)) \Rightarrow Q(x)))$
(d) $((\forall x(P(x) \Leftrightarrow (\forall x Q(x))) \Rightarrow (\forall x P(x)) \Leftrightarrow Q(x)))$

10. Consider the following first orde... relation symbol, $\forall x \forall y (R(x, y) \Rightarrow$

The formula is

(a) Satisfiable and valid
(c) Unsatisfiable but its negation
(d) Satisfiable but its negation i

11. Which one of the first order ... correctly express the following

"Tigers and Lions attack if the

(a) $\forall x[(\text{tiger}(x) \wedge \text{lion}(x)) \rightarrow [$
(b) $\forall x[(\text{tiger}(x) \vee \text{lion}(x)) \rightarrow [$
(c) $\forall x[(\text{tiger}(x) \vee \text{lion}(x)) \rightarrow$
(d) $\forall x[(\text{tiger}(x) \vee \text{lion}(x)) \rightarrow$

12. Which one of the following

(a) $\forall x(P(x) \Rightarrow Q(x)) \Rightarrow ((\forall x P$
(b) $\exists x(P(x) \vee Q(x)) \Rightarrow ((\exists x P$
(c) $\exists x(P(x) \wedge Q(x)) \Leftrightarrow ((\exists x P$
(d) $\forall x \exists y\, P(x, y) \Rightarrow \exists y \forall x P(x$

13. Which of the following fir... Here $\alpha(x)$ is a first order f... order formula with no fre...

(a) $[\beta \rightarrow (\exists x, \alpha(x))] \rightarrow [\forall x$
(c) $[(\exists x, \alpha(x)) \rightarrow \beta] \rightarrow [\forall x$

14. Which one of the follow... represent the statement: The following notations ... G(x) : x is gold ornamen... S(x) : x is a silver ornam... P(x) : x is precious

(a) $\forall x(P(x) \rightarrow (G(x) \wedge S($
(c) $\exists x((G(x) \wedge S(x)) \rightarrow P$

15. What is the correct ... mathematical logic? "Some real numbers a...

(a) $\exists\, x$ (real $(x) \vee$ ration
(c) $\exists\, x$ (real $(x) \wedge$ ratio

16. What is the logical tra... "None of my friends a...

(a) $\exists x\, (F(x) \wedge \sim P(x))$
(c) $\exists x\, (\sim F(x) \wedge \sim P(x))$

10. **Consider the following first order logic formula in which R is a binary relation symbol, $\forall x \forall y (R(x, y) \Rightarrow R(y, x)$**
The formula is
(a) Satisfiable and valid
(b) Satisfiable and so is its negation
(c) Unsatisfiable but its negation is valid
(d) Satisfiable but its negation is unsatisfiable

11. **Which one of the first order predicate calculus statements given below correctly express the following English statement ?**
"Tigers and Lions attack if they are hungry or threatened"
(a) $\forall x[(\text{tiger}(x) \wedge \text{lion}(x)) \rightarrow [\{\text{hungry}(x) \vee \text{threatened}(x)) \rightarrow \text{attacks}(x)]]$
(b) $\forall x[(\text{tiger}(x) \vee \text{lion}(x)) \rightarrow [\{\text{hungry}(x) \vee \text{threatened}(x)) \wedge \text{attacks}(x)]]$
(c) $\forall x[(\text{tiger}(x) \vee \text{lion}(x)) \rightarrow \{\{\text{attack}(x) \rightarrow \text{hungry}(x) \vee \text{threatened}(x)]]$
(d) $\forall x[(\text{tiger}(x) \vee \text{lion}(x)) \rightarrow \{\{\text{hungry}(x) \vee \text{threatened}(x)) \rightarrow \text{attacks}(x)]]$

12. **Which one of the following first order logic formulae is valid?**
(a) $\forall x(P(x) \Rightarrow Q(x)) \Rightarrow ((\forall x P(x)) \Rightarrow (\forall x Q(x)))$
(b) $\exists x(P(x) \vee Q(x)) \Rightarrow ((\exists x\, P(x)) \Rightarrow (xQ(x)))$
(c) $\exists x(P(x) \wedge Q(x)) \Leftrightarrow ((\exists x\, P(x)) \wedge (\exists x Q(x)))$
(d) $\forall x \exists y\, P(x, y) \Rightarrow \exists y \forall x P(x, y)$

13. **Which of the following first order formulae is logically valid?**
Here $\alpha(x)$ is a first order formulae with x as a free variable and β is a first order formula with no free variable.
(a) $[\beta \rightarrow (\exists x, \alpha(x))] \rightarrow [\forall x, \beta \rightarrow \alpha(x)]$
(b) $[\exists x, \beta \rightarrow \alpha(x)] \rightarrow [\beta \rightarrow (\forall x\, \alpha(x))]$
(c) $[(\exists x, \alpha(x)) \rightarrow \beta] \rightarrow [\forall x, \alpha(x) \rightarrow \beta]$
(d) $[(\forall x, \alpha(x)) \rightarrow \beta] \rightarrow [\forall x\, \alpha(x) \rightarrow \beta]$

14. **Which one of the following is the most appropriate logical formula to represent the statement: "Gold and silver ornaments are precious"**
The following notations are used :
$G(x)$: x is gold ornament
$S(x)$: x is a silver ornament
$P(x)$: x is precious
(a) $\forall x(P(x) \rightarrow (G(x) \wedge S(x)))$
(b) $\forall x(G(x) \wedge (S(x) \rightarrow P(x)))$
(c) $\exists x((G(x) \wedge S(x)) \rightarrow P(x)))$
(d) $\forall x((G(x) \vee S(x)) \rightarrow P(x))$

15. **What is the correct translation of the following statement into mathematical logic?**
"Some real numbers are rational"
(a) $\exists\, x$ (real $(x) \vee$ rational (x))
(b) $\forall x$ (real $(x) \rightarrow$ rational (x))
(c) $\exists\, x$ (real $(x) \wedge$ rational (x))
(d) $\exists x$ (rational $(x) \rightarrow$ real (x))

16. **What is the logical translation of the following statements?**
"None of my friends are perfect"
(a) $\exists x\, (F(x) \wedge \sim P(x))$
(b) $\exists x\, (\sim F(x) \wedge P(x))$
(c) $\exists x\, (\sim F(x) \wedge \sim P(x))$
(d) $\sim \exists x\, (F(x) \wedge P(x))$

17. **Which one of the following is NOT logically equivalent to $\neg\exists x(\forall y(\alpha) \wedge \forall z(\beta))$?**
(a) $\forall x(\exists z(\neg\beta) \rightarrow \forall y(\alpha))$
(b) $\forall x(\forall z(\beta) \rightarrow \exists y(\neg\alpha))$
(c) $\forall x(\forall y(\alpha) \rightarrow \exists z(\neg\beta))$
(d) $\forall x(\exists y(\neg\alpha) \vee \exists z(\neg\beta)))$

18. **Consider the statement**
"Not all that glitters is gold"
Predicate glitters (x) is true if x glitters and predicate gold(x) is true if x is gold. Which one of the following logical formulae represents the above statement?
(a) $\forall x$: glitters(x) $\Rightarrow$ ~gold(x)
(b) $\forall x$: gold (x) $\Rightarrow$ glitters(x)
(c) $\exists x$: gold(x) $\wedge$ ~glitters(x)
(d) $\exists x$: glitters(x) $\wedge$ ~gold(x)

19. **Which of the following well-formed formulae is a tautology?**
(a) $\forall x \exists y\, R(x,y) \leftrightarrow \exists y \forall x\, R(x,y)$
(b) $(\forall x\, [\exists y\, R(x,y) \rightarrow S(x,y)]) \rightarrow \forall x \exists y\, S(x,y)$
(c) $[\forall x \exists y\, (P(x,y) \rightarrow R(x,y))] \leftrightarrow [\forall x \exists y (\neg P(x,y) \vee R(x,y))]$
(d) $\forall x\, \forall y\, P(x,y) \rightarrow \forall x \forall y\, P(x,y)$

20. **Which one of the following well formed formulae in predicate calculus is NOT valid?**
(a) $(\forall x\, p(x) \Rightarrow \forall x\, q(x)) \Rightarrow (\exists x \neg p(x) \vee \forall x\, q(x))$
(b) $(\exists x\, p(x) \vee \exists x\, q(x)) \Rightarrow \exists x(p(x) \vee q(x))$
(c) $\exists x\, p(x) \wedge q(x)) \Rightarrow (\exists x(p(x) \wedge \exists x\, q(x))$
(d) $\forall x\, (p(x) \vee q(x)) \Rightarrow (\forall x\, p(x) \vee \forall x\, q(x))$

21. **Consider the first order logic sentence F : $\forall x\, (\exists y\, R(x, y))$**
Assuming non-empty logical domains, which of the sentence below are implied by F?
(i) $\exists y\, (\exists x\, R(x,y))$ (ii) $\exists y\, (\forall x\, R(x,y))$ (iii) $\forall y\, (\exists x\, R(x,y))$ (iv) $\neg\exists x(\forall y\, R(x,y))$
(a) (iv) only
(b) (i) and (iv) only
(c) (ii) only
(d) (ii) and (iii) only

22. **First Order Logic is also known as ———**
(a) First Order Predicate Calculus
(b) Quantification Theory
(c) Lower Order Calculus
(d) All of the mentioned

23. **Which is created by using single propositional symbol?**
(a) Complex sentences
(b) Atomic sentences
(c) Composition sentences
(d) Note of the mentioned

24. **A predicate is a proposition containing———**
(a) Statics
(b) Variables
(c) Numbers
(d) None

25. **Predicates represent one or more variables that are determined on a specific ———**
(a) Domain
(b) Co-domain
(c) Both a and b
(d) None of these

26. **A ——— quantifies a variable of a**
(a) Proposition
(b) Quantity

27. **Which of the following is/are the**
(a) Existential
(c) Both a and b

28. **An existential quantifier can be**
(a) $(\exists x \in A)\, p(x)$
(c) $(\exists x)\, p(x)$

29. **In case of universal quantifer denoted by ———**
(a) $x \forall p(x)$
(b) $p(x), \exists$

30. **Which of the following state**
(a) An existentially quantifie sally quantified propositi
(b) An universally quantifie tially quantified proposit
(c) Both a and b
(d) None of the above

31. **What is the rule for the ne**
(a) Dissociative law
(c) De Morgan's law

32. **Multiple quantifiers can variable**
(a) One
(c) More than one

33. **——— to arange the quantifiers in a particu meaningful.**
(a) It is necessary
(c) Sometimes it is neces

A. Short and Long Answer Ty

1. (i) $\exists x\, (P(x) \wedge Q(x))$ (ii) $\forall$
(iii) $\exists x\, (P(x) \vee \neg Q(x))$ (iv)

2. (i) Every student of the sc
(ii) There is a student w Bengali.

26. A ——— quantifies a variable of a predicate
(a) Proposition (b) Quantity (c) Quality (d) Quantifier

27. Which of the following is/are the type(s) of quantifier in predicate logic?
(a) Existential (b) Universal
(c) Both a and b (d) None of the above

28. An existential quantifier can be written in which way(s) in a proposition?
(a) $(\exists x \in A)p(x)$ (b) $(\exists x \in A)$such that $p(x)$
(c) $(\exists x)p(x)$ (d) all of the above

29. In case of universal quantifer, the proposition $p(x)$ over the universe U is denoted by ———
(a) $x \forall p(x)$ (b) $p(x), \exists \forall$ (c) $p(x), x \forall$ (d) $\forall x, p(x)$

30. Which of the following statement is/are TRUE?
(a) An existentially quantified proposition arises from negating a universally quantified proposition
(b) An universally quantified proposition arises from negating a existentially quantified proposition
(c) Both a and b
(d) None of the above

31. What is the rule for the negation of quantified proposition?
(a) Dissociative law (b) Associative law
(c) De Morgan's law (d) Identity law

32. Multiple quantifiers can be used to quantify propositions with ——— variable
(a) One (b) Two
(c) More than one (d) None

33. ——— to arange the multiple universal quantifiers ro existential quantifiers in a particular order in order to make the proposition meaningful.
(a) It is necessary (b) It is not necessary
(c) Sometimes it is necessary (d) none of the above

Answers

A. Short and Long Answer Type Questions

1. (i) $\exists x\,(P(x) \wedge Q(x))$ (ii) $\forall x \sim (P(x) \vee Q(x))$
(iii) $\exists x\,(P(x) \vee \sim Q(x))$ (iv) $\forall x(P(x) \wedge Q(x))$

2. (i) Every student of the school either can speak English or know Bengali
(ii) There is a student who cannot speak English and also do not know Bengali.

(iii) No student in the school can speak English and knows Bengali.
(iv) All the students in the school can speak English but do not know Ben

3. (i) There exists an animal which does not live in water.
(ii) There exists a fish that is not a shark.
(iii) Every shark that lives in water is a fish.
(iv) Every shark which is a fish lives in water.

4. (i) $(\forall x \in D)M(x)$, where $M(x)$: "x is mortal"
(ii) $-(\forall n \in N)P(n)$, where $P(n)$: "$n + 5 < 10$"
(iii) $-\forall x(P(x) \to Q(x))$, where $P(x)$: "x is a person" $Q(x)$: "x is rich"
(iv) $\forall x(P(x) \to (Q(x) \vee R(x)))$, where $P(x)$: "x is natural number."
$Q(x)$: "x is even" $R(x)$: "x is odd"
(v) $(\exists x)(P(x) \wedge Q(x))$, where $P(x)$: "x is even". $Q(x)$: "x is prime"
(vi) $(\forall x)(\exists y)P(x,y)$ or $\forall x \exists y\ (xy = 0)$, where $P(x,y)$: "$xy = 0$"
(vii) $(\forall x)(P(x) \leftrightarrow (Q(x) \wedge R(x)))$, where $P(x)$: "x is divisible by 6".
$Q(x)$: "x is divisible by 2". $R(x)$: "x is divisible by 3"

12. (i) Valid. (ii) Valid. (iii) Invalid. (iv) Invalid.

13. (i) Valid. (ii) Valid. (iii) Valid.

16. (i) False. (ii) True. (iii) True. (iv) False. (v) True.

17. (i) True. (ii) True.

18. (i) True. (ii) False.

21. (i) False. (ii) True. (iii) True. (iv)True.

25. (i) False. (ii) True. (iii)True.

27. (i) Not every student in this class is intelligent.
(ii) Some student in this class is not intelligent.

28. (i) $\exists$ a real number x such that $x > 3$ and $x^2 \leq 9$
(ii) $\exists$ a real number x such that $x\,(x + 1) > 0$ and $x \leq 0$, $x \geq -1$

29. (i) $(\forall x) - P(x) \vee (\forall y) - Q(y)$
(ii) $(\exists x) - P(x) \wedge (\forall y) - Q(y)$
(iii) $\exists x \exists y \forall z - P(x,y,z)$
(iv) $\forall x \exists y \forall z - P(x,y,z)$
(v) $\exists x \forall y - (P(x, y) \to Q(x, y))$
(vi) $\forall x \forall y (-P(x) \vee Q(y))$
(vii) $(\forall y) - (P(y) \to (\forall x) - Q(x))$

30. (i) $\exists x\ \exists y,\ P(x, y)$
(ii) $(\forall x \forall y\, R(x, y) \vee (\exists x \exists y, -P(x, y))$

B. Multiple Choice Questions (MCQ)

1. a	3. a	5. d	6. c	8. b	10. b	12. d	13. c	14. d	15.
16. c	17. a	18. d	20. d	21. b	22. d	23. b	24. b	25. a	26.
27. c	28. d	29. d	30. c	31. c	32. c	33. b			

04

Set T

Introduction

Set Theory is the true study of in
place prominent in human culture.
which mathematics takes place toda
foundation for the rest of mathemati
theory of sets that was originated in
Cantor. Since, the fundamentals of S
basic problems in the subject seem e
different from the history of most o
long process can usually be traced
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Set theory however is rather diff
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Set theory is one of the grea
Basically all mathematical concept
within axiomatic set theory. Thus
systematizing modern mathemati
questions about admissible mat
question of existence principles.
process by which set theory came
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4.1 Set

A well-defined collection of d
collection of group of members
set is commonly represented as
we have a set we say that some

04

Set Theory

Introduction

Set Theory is the true study of infinity. This alone assures the subject of a place prominent in human culture. But even more, Set Theory is the milieu in which mathematics takes place today. As such, it is expected to provide a firm foundation for the rest of mathematics. Most of Mathematics is based upon the theory of sets that was originated in 1870s by the German mathematician George Cantor. Since, the fundamentals of Set Theory are known to all mathematicians, basic problems in the subject seem elementary. The history of set theory is rather different from the history of most other areas of mathematics. For most areas a long process can usually be traced in which ideas evolve until an ultimate flash of inspiration, often by a number of mathematicians almost simultaneously, produces a discovery of major importance.

Set theory however is rather different. It is the creation of one person, George Cantor.

Set theory is one of the greatest achievements of modern mathematics. Basically all mathematical concepts, methods, and results admit of representation within axiomatic set theory. Thus set theory has served quite a unique role by systematizing modern mathematics, and approaching in a unified form all basic questions about admissible mathematical arguments—including the thorny question of existence principles. This entry covers in outline the convoluted process by which set theory came into being, covering roughly the years 1850 to 2000. We now provide a formal definition of set.

4.1 Set

A well-defined collection of distinct objects is called a set. It relates with the collection of group of members or elements in mathematics or in real world. A set is commonly represented as a list of all its members enclosed in braces. If we have a set we say that some objects belong (or do not belong) to this set, are

(or are not) in the set. We say also that sets consist of their elements repre... within the curly brackets { }. For example : {1, 2, 3, 4} is a set of numbers. ... {1, 2, 2, 3, 4} is not a set since here the number 2 appears twice. It is called multi... The objects in the set are called elements or members of that set. The symbol ∈ indicates set membership and means "is an element of" so that the statement $x \in A$ means that x is an element of the set A. Here, $1 \in \{1, 2, 3, 4\}$. The negation of set membership is denoted by the symbol "∉". This means that "x is not an element of A" or "x does not belong to A". Here, $5 \notin \{1, 2, 3, 4\}$

The intuitive idea of a set is probably even older than that of number. The concept of set theory is important everywhere in mathematics because every field of mathematics uses or refers to sets in some way. They are important in building more complex mathematical structure.

If we consider the collection of people in a particular village who are honest. Here the term "honest" is not well defined. So, this collection of honest people in a village is not a set.

Let us consider the following collections of objects :

1. The natural numbers.
2. The trees in a garden.
3. The men in a town.

In each of the above collections, the objects have been combined together according to some rules. That is, these collections are well-defined. So, all the above collections are examples of sets.

In general, we denote a set by A, B, C.... and its elements by a, b, c.... If a is an element of A, we write $a \in A$ [i.e., a belongs to A] and if a be not an element of A, we write $a \notin A$.

4.2 Set Formation

Sets can be represented in three different forms. Let us discuss each of them in detail by taking examples.

1. Statement Form or Descriptive Form
2. Roster or Tabular Form
3. Rule or Set Builder Form

Let us discuss the forms.

1. **Statement Form :** In this Form, well-defined descriptions are provided for the elements or members in the set. Verbal Description of Elements is given. Statement Form is also known as Descriptive Form. For example :
   (i) Set of Odd Numbers Less than 10.
   (ii) Set of Students in Class V having a height above 5 ft.
   (iii) Set of Numbers greater than 30 and less than 40.

2. **Tabular form :** If a set is defined by actually listing the set A contains the elements p, q, r, s then $A = \{p, q, r, s\}$
For example :
(i) Set of Natural Numbers Less than 10
(ii) Set of Vowels in the Word Elephant
This is called Tabular form of set.

3. **Set Builder form :** A set said to be in set builder form, properties which its elements satisfy.
(i) $B = \{x : x \in N, x \text{ is a multiple of } 2\}$
(ii) Set Builder Form of $A = \{3, 6, 9, \ldots$ and $n < 15\}$

4.3 Some Standard Sets

Some standard notations in set theory

N = set of natural numbers.
Z = set of all integers = $\{0, \pm1, \pm2$
$Z^+ = \{1, 2, 3, \ldots\}$
$Z^- = \{-1, -2, -3, \ldots\}$
Q = set of rational numbers.
Q^+ = set of all positive rational
R = set of all real numbers.
R^+ = set of positive real numb
C = set of complex numbers.

Some important definitions rela

4.4 Types of Sets

1. Subset

Let S be a set. A set T is said to
We denote this $T \subseteq S$. If $T \subset S$

Example 4.1 : Let us conside
Then B is a subset of A.

2. Proper subset

If B be a proper subset of A, least one element which does

Example 4.2 : Let us consid
26 letters. Now the set of

2. **Tabular form :** If a set is defined by actually listing its members, such as if the set A contains the elements p, q, r, s then it is expressed as
$A = \{p, q, r, s\}$
For example :
(i) Set of Natural Numbers Less than 10 is {1, 2, 3, 4, 5, 6, 7, 8, 9}
(ii) Set of Vowels in the Word Elephant is W = {E, A}
This is called Tabular form of set.

3. **Set Builder form :** A set said to be in set builder form if it is defined by the properties which its elements satisfy. For example
(i) $B = \{x : x \in N, x \text{ is a multiple of } 2\}$
(ii) Set Builder Form of A = {3, 6, 9, 12} is $A = \{x : x = \text{multiples of } 3, n \in N \text{ and } n < 15\}$

4.3 Some Standard Sets

Some standard notations in set theory are :
N = set of natural numbers.
$\mathbb{Z}$ = set of all integers = {0, ±1, ±2, ...}
$\mathbb{Z}^+$ = {1, 2, 3, ...}
$\mathbb{Z}^-$ = {–1, –2, –3, ...}
Q = set of rational numbers.
Q^+ = set of all positive rational numbers.
R = set of all real numbers.
R^+ = set of positive real numbers.
C = set of complex numbers.
Some important definitions related to set are as follows :

4.4 Types of Sets

1. Subset

Let S be a set. A set T is said to be a subset of S if $x \in T \Rightarrow x \in S$.
We denote this $T \subseteq S$. If $T \subset S$, S is said to be a super set of T.

Example 4.1 : Let us consider the set A = {1, 2, 3, 4, 5, 6}. If we take B = {2, 4, 6}. Then B is a subset of A.

2. Proper subset

If B be a proper subset of A, then all elements of B belongs to A, but A has at least one element which does not belongs to B. We denote this $B \subset A$.

Example 4.2 : Let us consider the set of English alphabet. Here the set contains 26 letters. Now the set of vowel will be the proper subset of English alphabet.

3. Improper Subset

A subset which contains all the elements of the original set is called an Improper subset. It is denoted by $\subseteq$.

Example 4.3 : A = {1, 2, 3}. If we take B = {1, 2, 3}. Then A is improper subset of B.

Example 4.4 : Every set is a improper subset of itself.

4. Null set or empty set

A set consisting of no element is called null set or empty set.

We denote a null set by $\varnothing$.

Null set is a subset of any set. The sets A, $\varnothing$ are improper subsets of A. All other subset of A are called proper subset of A. Sometime it is known as empty set or void set.

Example 4.5 : Consider the set as the men or women whose height is more than 15ft. Here the set is void.

Example 4.6 : A = $\{x : x^2 = 4, x \text{ is odd}\}$. A is a null set.

Note

The set $\varnothing = \{0\}$ is not a null set because 0 is the element.

5. Finite set

A set is said to be finite set if either it is empty or it contains a finite number of elements.

Example 4.7 : Set of vowel is a finite set since it contains only five elements.

Example 4.8 : A = $\{x; x \in N, 3 < x < 6\}$

6. Infinite set

A set is said to be infinite set if the number of elements is unlimited in it.

Example 4.9 : Set of all points in a line segment is an infinite set.

Example 4.10 : B = $\{x; x \in N, x \text{ is a multiple of } 3\}$ is an infinite set.

7. Equality of two sets

Two set A and B is said to be equal if A is a subset of B and B is a subset of A. We denote it by A = B.

Example 4.11 : A = {1, 2, 3, 4, 5, 6} and B = $\{x : 1 \le x \le 6$: where x is a natural no.$\}$ Here, the two sets A and B are two equal sets.

8. Power set

The collection of all subsets of a set A is by P(A).

Example 4.12 : Let us consider set X the all possible subsets of X.

Mathematically it will be written as

9. Cardinality of a set

The cardinality of a set is defined set.

Let P = {2, 4, 6, 8, 10}. Now cardin

Note

If a set contains 'n' no elements th

10. Universal set

If all sets under investigation called Universal set.

Example 4.13 : In plane geor in the plane

4.5 Venn diagrams

Venn diagram also called representation of sets. It is us relation between the set ope union, intersection, complem of a set. For example A $\subset$ B (Venn diagram as follows :

8. Power set

The collection of all subsets of a set A is called the power set of A, is denoted by P(A).

Example 4.12 : Let us consider set X = {1, 2, 3}. Now Power set of X will be as the all possible subsets of X.

Mathematically it will be written as P(X) = {∅, {1}, {2}, {3}, {1, 2}, {2, 3}, {1, 3}, X}

9. Cardinality of a set

The cardinality of a set is defined as the number of elements contained in a set.

Let P = {2, 4, 6, 8, 10}. Now cardinality of P is 5. Denoted by $n(P)$, so $n(P) = 5$

Note

If a set contains 'n' no elements then cardinality of power set will be $n(S) = 2^n$.

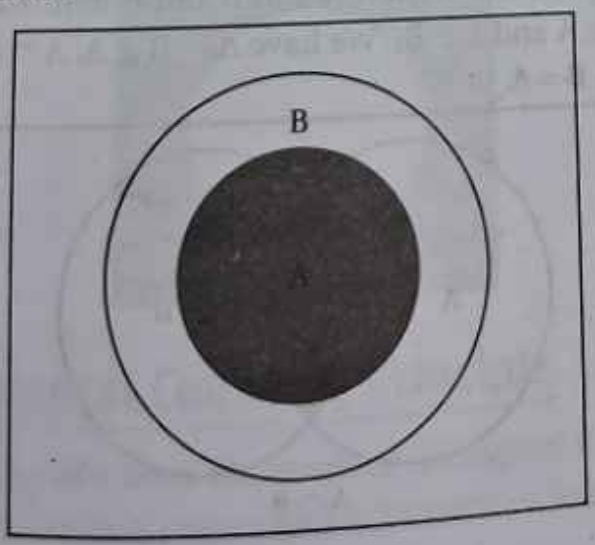
10. Universal set

If all sets under investigation are subsets of a fixed set U, then the set U is called Universal set.

Example 4.13 : In plane geometry, the Universal set consists of all the points in the plane

4.5 Venn diagrams of Sets

Venn diagram also called set diagram or a logic diagram is pictorial representation of sets. It is used to describe different types of set operations, relation between the set operations and operations performed on them like union, intersection, complement, difference etc. It is also used to depict subsets of a set. For example A ⊂ B (A is a proper subset of B) can be expressed using Venn diagram as follows :



4.6 Algebraic Operations on Sets

The following are some algebraic operations on set

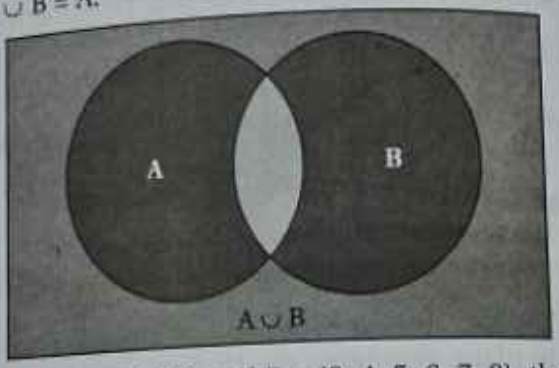
1. Union

Let A and B be two subsets of a universal set U. The union of A and B denoted by $A \cup B$ is the collection of all those elements that either belong to A or B and is defined by $A \cup B = \{x : x \in A \text{ or } x \in B\}$.

$x \in A \cup B \Rightarrow x$ belongs to at least one of the two sets A and B. We have

$A \subseteq A \cup B$ and $B \subseteq A \cup B$

If $B \subseteq A$, then $A \cup B = A$.



Example 4.14 : For $A = \{1, 2, 3\}$ and $B = \{3, 4, 5, 6, 7, 8\}$, then

$A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8\}$

Note

Union also means join or logical sum

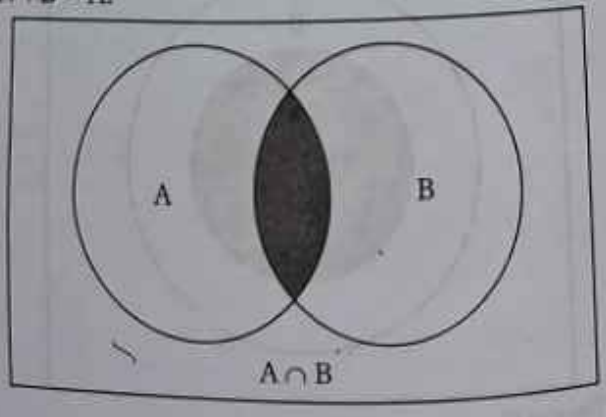
2. Intersection

Let A and B be two subset of a universal set U.

Intersection of two sets A and B denoted by $A \cap B$ is the collection of all those elements that belong to both the sets A and B and is defined by

$A \cap B = \{x : x \in A \text{ and } x \in B\}$. We have $A \cap B \subseteq A$, $A \cap B \subseteq B$

If $A \subseteq B$, then $A \cap B = A$.



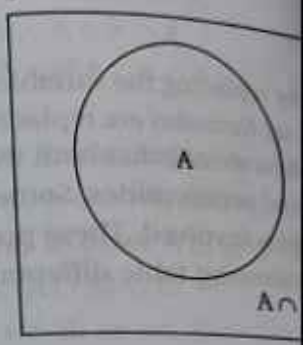
Example 4.15 : For $A = \{1, 2, 3, 4\}$ and

Notes

1. $x \notin A \cup B \Rightarrow x \notin A$ and $x \notin B$.
   $x \notin A \cap B \Rightarrow x \notin A$ or $x \notin B$.
2. Intersection is also called meet

3. Disjoint sets

Let A and B be two subset of a universal
If two subsets A and B have no common
B are said to be disjoint.



Example 4.16 : The sets $A = \{1, 2, 3,$
have no element in common

4. Complementation

The complement of a set A is a sub
$A' = \{x \in U : x \notin A\}$

Example 4.17 : Let U be the set o
$A' = \{$all natural numbers except

4.7 Fundamental Cou

The term cardinality of a set is
et. Cardinality can be finite (a no

Example 4.15 : For A = {1, 2, 3, 4} and B = {3, 4, 5, 6, 7, 8}, then A ∩ B = {3, 4}

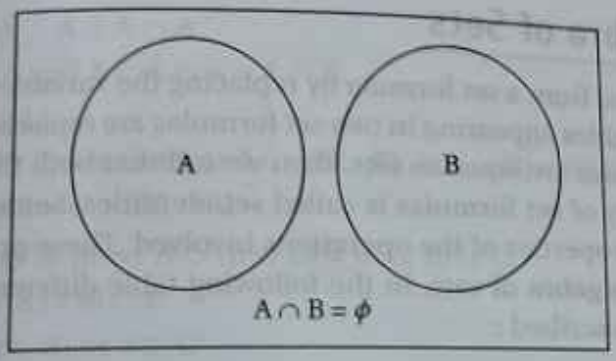
Notes

1. $x \notin A \cup B \Rightarrow x \notin A$ and $x \notin B$.
   $x \notin A \cap B \Rightarrow x \notin A$ or $x \notin B$.
2. Intersection is also called meet

3. Disjoint sets

Let A and B be two subset of a universal set U.

If two subsets A and B have no common element, then $A \cap B = \varnothing$ and A and B are said to be disjoint.



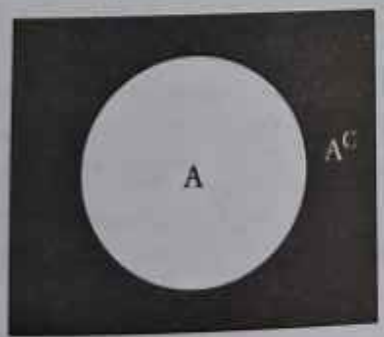
Example 4.16 : The sets A = {1, 2, 3, 4} and B = {5, 6, 7, 8, 9} are disjoint as they have no element in common

4. Complementation

The complement of a set A is a subset of U, denoted by A′ or A^C and defined by

$A' = \{x \in U : x \notin A\}$

Example 4.17 : Let U be the set of all natural numbers and A = {1, 2, 3, 4} then A′ = {all natural numbers except 1, 2, 3, 4}



4.7 Fundamental Counting Principle

The term cardinality of a set is the number of cardinal (basic) members in the set. Cardinality can be finite (a non-negative integer) or infinite. For example, the

set A = {1, 2, 6} has a cardinality of 3 since the set A has three elements in it. The cardinality of the set of integers is denumerably infinite. Following theorems for cardinality of sets :

Theorem-4.1 : If A and B are finite sets, then $A \cup B$ and $A \cap B$ are also finite and $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

where 'n' is the cardinal number.

Theorem-4.2 : If A and B are finite disjoint sets, then

$$n(A \cup B) = n(A) + n(B)$$

Theorem-4.3 : If A, B, C are finite sets then

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$$

4.8 Algebra of Sets

A set is obtained from a set formula by replacing the variables by definite sets. When the set variables appearing in two set formulas are replaced by any sets and both the set formulas are equal as sets, then we call that both the set formulas are equal. The equality of set formulas is called set identities. Some of the identities describe certain properties of the operations involved. These properties describe an algebra called algebra of sets. In the following table different laws of algebra of set theory are described :

Table 4.1

(i) Idempotent Laws	(vi) Identity Laws
(a) $A \cup A = A$	(a) $A \cup \phi = A$
(b) $A \cap A = A$	(b) $A \cap \phi = \phi$
(ii) Associative Laws	(c) $A \cup U = U$
(a) $A \cup (B \cup C) = (A \cup B) \cup C$	(d) $A \cap U = A$
(b) $A \cap (B \cap C) = (A \cap B) \cap C$	(vii) Complement Laws
(iii) Commutative Laws	(a) $A \cup A' = U$
(a) $A \cup B = B \cup C$	(b) $A \cap A' = \phi$
(b) $A \cap B = B \cap A$	(c) $U' = \phi$
(iv) Distributive Laws	(d) $\phi' = U$
(a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup B)$	(viii) Involution Laws
(b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	(a) $(A')' = A$
(v) De Morgan's Laws	(ix) Absorptive Laws
(a) $(A \cup B)' = A' \cap B'$	$A \cup (A \cap B) = A$
(b) $(A \cap B)' = A' \cup B'$	$A \cap (A \cup B) = A$

Note

In the laws of Algebra of sets involving $\cup, \cap, \varnothing, S$, if we replace $\cup$ by $\cap$; $\cap$ by $\cup$; $\varnothing$ by S and S by $\varnothing$, then we obtain another law of sets and this law is valid law which is kown as Principle of Duality.

Example 4.18 : Prove that (a

Solution : (a) To prove $A \cup$

Since $B \subset A \cup B$ therefore A

Let $x \in A \cup A \Rightarrow x \in A$ or, x

$\therefore A \cup A \subset A$

As $A \cup A \subset A$ and $A \subset A \cup$

Hence proved.

(b) To prove $A \cap A = A$

Since, $A \cap B \subset B$, therefo

Let $x \in A \Rightarrow x \in A$ and x

$\Rightarrow x \in A \cap A \therefore A \subset A \cap$

As $A \subset A \cap A$ and $A \subset A$

Hence proved.

Example 4.19 : Prove tha

(b) $A \cap (B \cap C) = (A \cap B$

Solution : (a) To prove

Let some $x \in (A \cup B) \cup$

$\Rightarrow (x \in A$ or $x \in B)$ or x

$\Rightarrow x \in A$ or $x \in B$ or $x \in$

$\Rightarrow x \in A$ or $(x \in B$ or x

$\Rightarrow x \in A$ or $x \in B \cup C$

$\Rightarrow x \in A \cup (B \cup C)$

Similarly, if some $x \in A$

Thus, any $x \in A \cup (B \cup$

Hence proved.

(b) To prove $A \cap (B \cap C)$

Let some $x \in A \cap (B$

$\Rightarrow x \in A$ and $(x \in B$ a

$\Rightarrow (x \in A$ and $x \in B)$

$\Rightarrow x \in (A \cap B) \cap C$

Similarly, if some $x \in$

Thus, any $x \in A \cap (B$

Hence proved.

Example 4.20 : Prove

Solution : (a) To pro

$A \cup B = \{x : x \in A$ or x

$= \{x : x \in B$ or x

Hence proved.

Example 4.18 : Prove that (a) $A \cup A = A$ and (b) $A \cap A = A$.

Solution : (a) To prove $A \cup A = A$

Since $B \subset A \cup B$ therefore $A \subset A \cup A$

Let $x \in A \cup A \Rightarrow x \in A$ or, $x \in A \Rightarrow x \in A$

$\therefore A \cup A \subset A$

As $A \cup A \subset A$ and $A \subset A \cup A \Rightarrow A = A \cup A$

Hence proved.

(b) To prove $A \cap A = A$

Since, $A \cap B \subset B$, therefore $A \cap A \subset A$

Let $x \in A \Rightarrow x \in A$ and $x \in A$

$\Rightarrow x \in A \cap A \therefore A \subset A \cap A$

As $A \subset A \cap A$ and $A \subset A \cap A = A \cap A$.

Hence proved.

Example 4.19 : Prove that (a) $A \cup (B \cup C) = (A \cup B) \cup C$ and (b) $A \cap (B \cap C) = (A \cap B) \cap C$

Solution : (a) To prove $A \cup (B \cup C) = (A \cup B) \cup C$

Let some $x \in (A \cup B) \cup C$

$\Rightarrow (x \in A$ or $x \in B)$ or $x \in C$

$\Rightarrow x \in A$ or $x \in B$ or $x \in C$

$\Rightarrow x \in A$ or $(x \in B$ or $x \in C)$

$\Rightarrow x \in A$ or $x \in B \cup C$

$\Rightarrow x \in A \cup (B \cup C)$

Similarly, if some $x \in A \cup (B \cup C)$, then $x \in (A \cup B) \cup C$.

Thus, any $x \in A \cup (B \cup C) \Leftrightarrow x \in (A \cup B) \cup C$

Hence proved.

(b) To prove $A \cap (B \cap C) = (A \cap B) \cap C$

Let some $x \in A \cap (B \cap C) \Rightarrow x \in A$ and $x \in B \cap C$

$\Rightarrow x \in A$ and $(x \in B$ and $x \in C) \Rightarrow x \in A$ and $x \in C$

$\Rightarrow (x \in A$ and $x \in B)$ and $x \in C \Rightarrow x \in A \cap B$ and $x \in C$

$\Rightarrow x \in (A \cap B) \cap C$

Similarly, if some $x \in A \cap (B \cap C)$, then $x \in (A \cap B) \cap C$

Thus, any $x \in A \cap (B \cap C) \Leftrightarrow x \in (A \cap B) \cap C$

Hence proved.

Example 4.20 : Prove that (a) $A \cup B = B \cup A$ (b) $A \cap B = B \cap A$.

Solution : (a) To prove $A \cup B = B \cup A$

$A \cup B = \{x : x \in A$ or $x \in B\}$

$= \{x : x \in B$ or $x \in A\}$ ($\because$ Order is not preserved in case of sets)

Hence proved.

(b) To prove $A \cap B = B \cap A$

$A \cap B = \{x : x \in A \text{ and } x \in B\}$

$= \{x : x \in B \text{ and } x \in A\}$ {$\because$ Order is not preserved in case of sets}

$= B \cap A$.

Hence proved.

Example 4.21 : Prove that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Solution : Let $P = A \cap (B \cup C)$ and $Q = (A \cap B) \cup (A \cap C)$.

Let x be an arbitrary element of P.

$x \in A \cap (B \cup C)$

$\Rightarrow x \in A$ and $x \in B \cup C$

$\Rightarrow x \in A$ and $\{x \in B$ or $x \in C\}$

$\Rightarrow \{x \in A$ and $x \in B\}$ or $\{x \in A$ and $x \in C\}$

$\Rightarrow x \in A \cap B$ or $x \in A \cap C$

$\Rightarrow x \in (A \cap B) \cup (A \cap C)$

$\Rightarrow x \in Q$

$\therefore x \in P \Rightarrow x \in Q$

$\therefore P \subseteq Q$

Let y be an arbitrary element of Q.

$y \in Q \Rightarrow y \in (A \cap B) \cup (A \cap C)$

$\Rightarrow y \in A \cap B$ or $y \in A \cap C$

$\Rightarrow \{y \in A$ and $y \in B\}$ or $\{y \in A$ and $y \in C\}$

$\Rightarrow y \in A$ and $\{y \in B$ or $y \in C\}$

$\Rightarrow y \in A$ and $y \in B \cup C$

$\Rightarrow y \in A \cap (B \cup C)$

$\Rightarrow y \in P$

$\therefore y \in Q \Rightarrow y \in P$

$\therefore Q \subseteq P$

$\therefore$ From (1) and (2), we get, $P = Q$.

Hence proved.

Example 4.22 : Prove that $(A \cap B)^c = A^c \cup B^c$

Solution : Let $P = (A \cap B)^c$ and $Q = (A^c \cup B^c)$

Let x be an arbitrary element of p.

$x \in P \Rightarrow x \in (A \cap B)$

$\Rightarrow x \notin A \cap B$

$\Rightarrow x \notin A$ or $x \notin B$

$\Rightarrow x \in A^c$ or $x \in B^c$

$\Rightarrow x \in A^c \cup B^c$

$\Rightarrow x \in Q$

$\therefore x \in P \Rightarrow x \in Q$

$\therefore P \subseteq Q$

Let y be an arbitrary elem

$y \in Q \Rightarrow y \in (A^c \cup B^c)$

$\Rightarrow y \in A^c$ or $y \in B^c$

$\Rightarrow y \notin A$ or $y \notin B$

$\Rightarrow y \notin A \cap B$

$\Rightarrow y \in (A \cap B)^c$

$\Rightarrow y \in P$

$y \in Q \Rightarrow y \in P$

$Q \subseteq P$

$\therefore$ From (1) and (2) we

Hence proved.

Example 4.23 : Prove

(d) $A \cap U = A$

Solution : (a) To pr

Let $x \in A \cup \phi \Rightarrow x \in$

$\Rightarrow x \in A$ ($\because x \notin \phi$ as

Therefore, $x \in A \cup$

Hence, $A \cup \phi \subset A$

We know that $A \subset A$

But for $B = \phi$, we h

From above, $A \subset A$

Hence proved.

(b) To prove $A \cap \phi =$

If $x \in A$, then $x \in$

Therefore, $x \in A$

Hence proved.

(c) To prove $A \cup U$

Every set is a su

$\therefore A \cup U \subseteq U$

Also, $U \subseteq A \cup U$

Therefore, $A \cup$

Hence proved.

(d) To prove $A \cap U$

We know $A \cap U$

So we have to

$\therefore x \in P \Rightarrow x \in Q$

$\therefore P \subseteq Q$ ... (1)

Let y be an arbitrary element of Q.

$y \in Q \Rightarrow y \in (A^c \cup B^c)$

$\Rightarrow y \in A^c$ or $y \in B^c$

$\Rightarrow y \notin A$ or $y \notin B$

$\Rightarrow y \notin A \cap B$

$\Rightarrow y \in (A \cap B)^c$

$\Rightarrow y \in P$

$y \in Q \Rightarrow y \in P$

$Q \subseteq P$ ... (2)

$\therefore$ From (1) and (2) we can write $P = Q$.

Hence proved.

Example 4.23 : Prove (a) $A \cup \phi = A$ (b) $A \cap \phi = \phi$ (c) $A \cup U = U$ (d) $A \cap U = A$

Solution : (a) To prove $A \cup \phi = A$

Let $x \in A \cup \phi \Rightarrow x \in A$ or $x \in \phi$

$\Rightarrow x \in A$ ($\because x \notin \phi$ as ϕ is the null set)

Therefore, $x \in A \cup \phi \Rightarrow x \in A$

Hence, $A \cup \phi \subset A$

We know that $A \subset A \cup B$ for any set B.

But for $B = \phi$, we have $A \subset A \cup \phi$

From above, $A \subset A \cup \phi$, $A \cup \phi \subset A \Rightarrow A = A \cup \phi$

Hence proved.

(b) To prove $A \cap \phi = \phi$

If $x \in A$, then $x \notin \phi$ ($\because \phi$ is null set)

Therefore, $x \in A$, $x \notin \phi \Rightarrow A \cap \phi = \phi$

Hence proved.

(c) To prove $A \cup U = U$

Every set is a subset of universal set.

$\therefore A \cup U \subseteq U$

Also, $U \subseteq A \cup U$

Therefore, $A \cup U = U$

Hence proved.

(d) To prove $A \cap U = A$ ... (1)

We know $A \cap U \subset A$

So we have to show that $A \subset A \cap U$

Let $x \in A \Rightarrow x \in A$ and $x \in U$ ($\because A \subset U$ so $x \in A \Rightarrow x \in U$)

$\therefore x \in A \Rightarrow x \in A \cap U$

$\therefore A \subset A \cap U$

From (1) and (2), we get $A \cap U = A$. Hence proved.

Example 4.24 : Prove that (a) $A \cup A' = U$ (b) $A \cap A' = \phi$ (c) $U' = \phi$ (d) ϕ'

Solution : To prove $A \cup A' = U$

Every set is a subset of Universal set U.

$\therefore A \cup A' \subset U$

We have to show that $U \subseteq A \cup A'$

Let $x \in U \Rightarrow x \in A$ or $x \notin A$

$\Rightarrow x \in A$ or $x \in A' \Rightarrow x \in A \cup A^c$

$\therefore U \subseteq A \cup A'$

From (1) and (2) we get, $A \cup A' = U$

Hence proved.

(b) To prove $A \cup A' = \phi$

As ϕ is a subset of every set

$\therefore \phi \subseteq A \cap A'$

We have to show that $A \cap A' \subseteq \phi$

Let $x \in A \cap A^c \Rightarrow x \in A$ and $x \in A^c$

$\Rightarrow x \in A$ and $x \notin A$ (A contradictory statement)

$\Rightarrow x \in \phi$

$\therefore A \cap A' \subseteq \phi$

From (1) and (2), we get $A \cap A' = \phi$

Hence proved.

(c) To prove $U' = \phi$

Let $x \in U' \Leftrightarrow x \notin U \Leftrightarrow x \in \phi$ (As U is a universal set)

$\therefore U' = \phi$

Hence proved.

Example 4.25 : Prove $(A')' = A$.

Solution : Let $x \in (A')' \Leftrightarrow x \notin A' \Leftrightarrow x \in A$

$\therefore (A')' = A$

Hence proved.

4.9 Difference of two sets

Let A and B be two subset of a universal set U.

The difference of two sets A and B, denoted by A – B is the collection of all elements in A that are not in B and is defined by

$A - B = \{x : x \in A \text{ and } x \notin B\}$

Also, we have $A - B = A \cap B'$

Example 4.26 : If A = {1, 2

4.9.1 Properties of

Some properties on differ

1. $A - (B \cap C) = (A - B)$
2. $A - (B \cup C) = (A - B)$

Example 4.27 : Prove th

Solution : $A - (B \cup C)$

$= A \cap B^c \cap C^c$ [Asso

$= A \cap A \cap B^c \cap C^c$

$= (A \cap B') \cap (A \cap$

$= (A - B) \cap (A - C)$

4.10 Symmetric

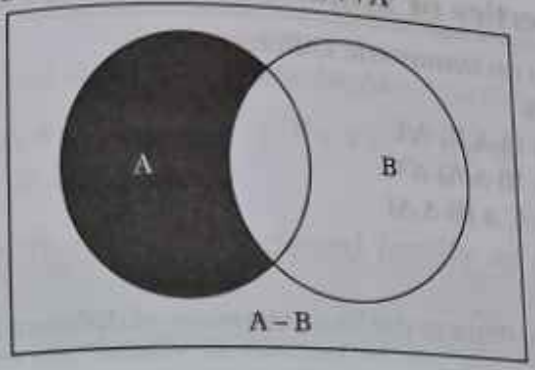
Let A and B be two sub

The symmetric differe collection of all elements written as $A \Delta B = (A - B)$

So, $A \Delta B = (A \cap B')$

Also, we have $A \Delta \phi$

Also, we have $A - B = A \cap B^c$ and $B - A = B \cap A^c$



Example 4.26 : If $A = \{1, 2, 5\}$ and $B = \{2, 5, 8\}$, then $A - B = \{1\}$ and $B - A = \{8\}$

4.9.1 Properties of Difference

Some properties on difference of sets are

1. $A - (B \cap C) = (A - B) \cup (A - C)$
2. $A - (B \cup C) = (A - B) \cap (A - C)$

Example 4.27 : Prove that $A - (B \cup C) = (A - B) \cap (A - C)$

Solution : $A - (B \cup C) = A \cap (B \cup C)^c = A \cap (B' \cap C')$ [by De Morgan's Law]

$= A \cap B^c \cap C^c$ [Associative prop.]

$= A \cap A \cap B^c \cap C^c$ [Idempotent prop.]

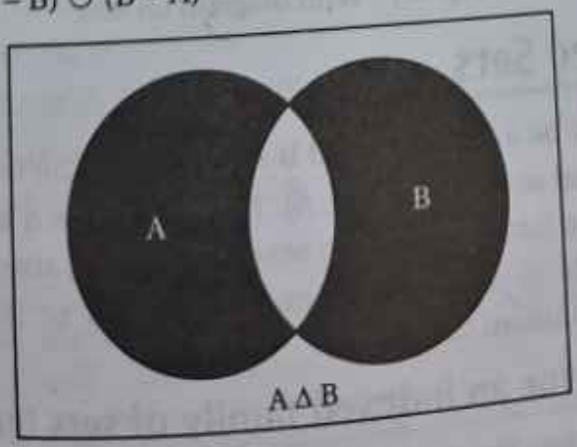
$= (A \cap B') \cap (A \cap C')$ [Commutative prop.]

$= (A - B) \cap (A - C)$ **[Proved].**

4.10 Symmetric Difference

Let A and B be two subset of a universal set U.

The symmetric difference of two subsets A and B denoted by $A \Delta B$ is the collection of all elements that belong either to A or to B but not to both and it is written as $A \Delta B = (A - B) \cup (B - A)$



So, $A \Delta B = (A \cap B') \cup (A' \cap B)$.

Also, we have $A \Delta \phi = A$ and $A \Delta A = \phi$.

4.10.1 Properties of Symmetric Difference

Some properties on Symmetric Difference are

1. $A \Delta B = B \Delta A$
2. $A \Delta (B \Delta C) = (A \Delta B) \Delta C$
   $= (B \Delta A) \Delta C$
   $= C \Delta (B \Delta A)$

Note

The following figure depicts the Venn diagram of different operations on sets

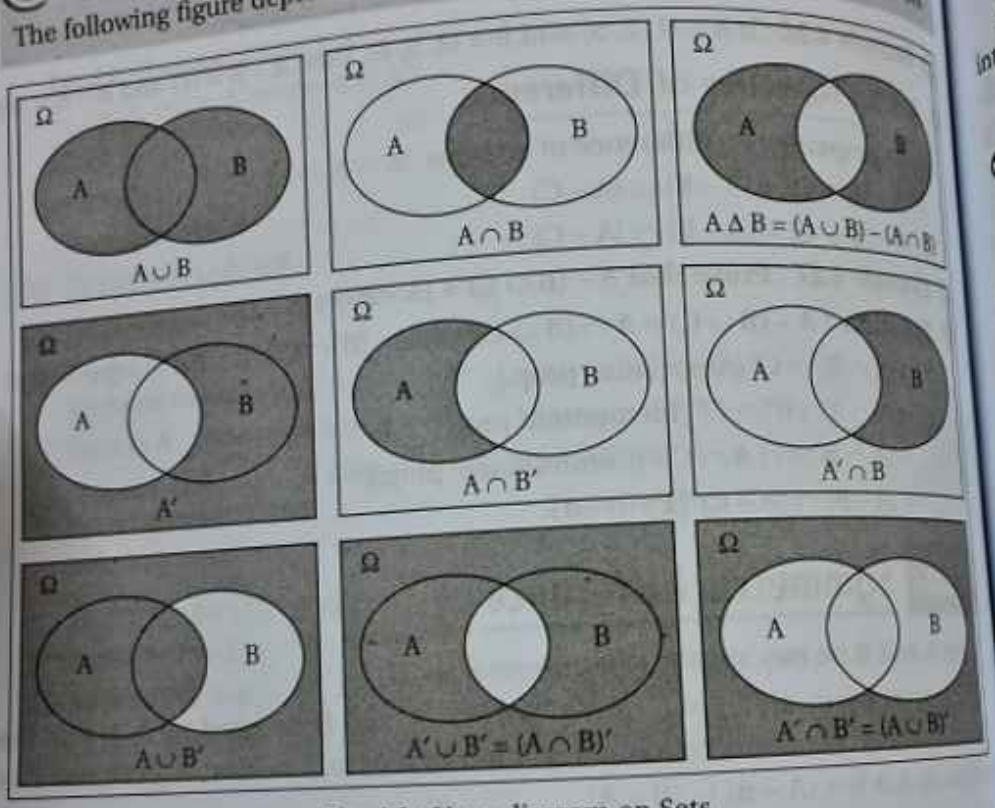


Fig. 4.1 : Venn diagram on Sets

4.11 Indexed Sets

Let $\wedge = \{1, 2, \ldots n\}$ be a finite set and $B = \{A_1, A_2, A_n\}$ be the family of n sets.

Let B be expressed as $B = \{A_\alpha : \alpha \in \Lambda\}$. In such a case Λ is called the index set or indexing set of the family B and the sets $A_1, A_2, \ldots A_n$ are called indexed sets.

An indexed family of sets A_i is denoted by $\{A_i\}_{i \in \Lambda}$ or $\{A_i : i \in \Lambda\}$. We call an element i of the domain an index.

4.11.1 Union for an indexed family of sets (generalized form)

Let $\{A_i : i \in \Lambda\}$ be any family of subsets of the universal set U. Then the union of the family of subsets is defined by $\bigcup_{i \in \Lambda} A_i = \{x : x \in A_i$ for atleast one $i \in \Lambda\}$

Example 4.28 : Let $\Lambda = \{1, 3, 4\}$ and A_i
$A_1 = \{1, 3, 5, 7\}$
Then $\bigcup_{i \in \Lambda} A_i = \{x : x \in A_i$ for at least on

Example 4.29 : Let $\Lambda = N$ and $A_n = \{x$
Then $\bigcup_{i \in \Lambda} A_i = \{x : 0 \le x \le 1\}$

4.11.2 Intersection for an in form)

Let $\{A_i : i \in \Lambda\}$ be any family of intersection of the family of subsets $\bigcap_{i \in \Lambda} A_i = \{x : x \in A_i$ for every $i \in$

Example 4.30 : Let $\Lambda = \{1, 3, 4\}$
$A_4 = \{1, 3, 5, 7\}$
$\bigcap_{i \in \Lambda} A_i = \{x : x \in A_i$ for every i

Example 4.31 : Let $\Lambda = N$ and
Then $\bigcap_{i \in N} A_i = \{0\}$

4.12 Sets of Set

If the elements of a set be s set of sets.

Example 4.32 : $A_n = \left\{-\frac{1}{n} <$

$\therefore A_1 = \{-1 < x < 1\}, A_2 = \{-$

Now we consider a set A
Here A is a set of sets.

Example 4.33 : $A = \{a, b$
Here subsets of A are ϕ,
$P(A) = \{\phi, \{a\}, \{b\}, \{a, b\}$
Here P(A) is called the

Note

The collection of all sub called the Power set of S then P(S) contains 2^n sub of n sets $A_1, A_2, \ldots, A_n$. T the Index set. Let I be t arbitrary collection of s

Example 4.28 : Let $\Lambda = \{1, 3, 4\}$ and $A_1 = \{2, 5\}$, $A_2 = \{2, 4, 5, 8\}$, $A_3 = \{2, 3, 5\}$, $A_4 = \{1, 3, 5, 7\}$

Then $\bigcup_{i\in\Lambda} A_i = \{x : x \in A_i \text{ for at least one } i \in \Lambda\} = A_1 \cup A_3 \cup A_4 = \{1, 2, 3, 5, 7\}$

Example 4.29 : Let $\Lambda = N$ and $A_n = \{x : 0 \le x \le \frac{1}{n}, n \in N\}$

Then $\bigcup_{i\in\Lambda} A_i = \{x : 0 \le x \le 1\}$

4.11.2 Intersection for an indexed family of sets (generalized form)

Let $\{A_i : i \in \Lambda\}$ be any family of subsets of the universal set U. Then the intersection of the family of subsets is defined by

$$\bigcap_{i\in\Lambda} A_i = \{x : x \in A_i \text{ for every } i \in \Lambda\}$$

Example 4.30 : Let $\Lambda = \{1, 3, 4\}$ and $A_1 = \{2, 5\}$, $A_2 = \{2, 4, 5, 8\}$, $A_3 = \{2, 3, 5\}$, $A_4 = \{1, 3, 5, 7\}$

$\bigcap_{i\in\Lambda} A_i = \{x : x \in A_i \text{ for every } i \in \Lambda\} = A_1 \cap A_3 \cap A_4 = \{5\}$

Example 4.31 : Let $\Lambda = N$ and $A_n = \{x : 0 \le x \le \frac{1}{n}, n \in N\}$

Then $\bigcap_{i\in N} A_i = \{0\}$

4.12 Sets of Set

If the elements of a set be sets themselves, then we have a family of sets or a set of sets.

Example 4.32 : $A_n = \left\{-\frac{1}{n} < x < \frac{1}{n} \mid n \in N\right\}$

$\therefore A_1 = \{-1 < x < 1\}$, $A_2 = \left\{-\frac{1}{2} < x < \frac{1}{2}\right\}$, $A_3 = \left\{-\frac{1}{3} < x < \frac{1}{3}\right\}$, ...

Now we consider a set A defined by $A = \{A_1, A_2, A_3, ...\} = \{A_\alpha \mid \alpha \in N\}$

Here A is a set of sets.

Example 4.33 : $A = \{a, b\}$

Here subsets of A are ϕ, $\{a\}$, $\{b\}$, $\{a, b\}$

$P(A) = \{\phi, \{a\}, \{b\}, \{a, b\}\}$

Here P(A) is called the power set of A.

Note

The collection of all subsets of nonempty sets of S is a set of sets. This set is called the Power set of S. and it is denoted by P(S). If S contains n elements, then P(S) contains 2^n subsets. Let I be a finite set $\{1, 2, ... n\}$ and F be the Family of sets $A_1, A_2, ..., A_n$. Then F can be expressed as $F = \{A_\alpha \mid \alpha \in I\}$. I is called the index set. Let I be the set N, then the family of sets $F = \{A_\alpha : \alpha \in I\}$ is an arbitrary collection of sets A_α, Indexed by I.

4.13 Partition of Sets

Let S be a non empty set. A family of non empty subsets $\{S_\alpha : \alpha \in I\}$, I the Index set, is said to form a Partition of S if

1. $\bigcup_{\alpha \in I} S_\alpha = S(S_1 \cup S_2 \cup S_3 \cup \ldots \cup S_n = S)$
2. $S_\alpha \cap S_\beta = \phi$ for $\alpha, \beta \in I$ and $\alpha \neq \beta$.

4.14 Cartesian Product of Sets

Let A and B be non empty sets. The Cartesian product of A and B, denoted by $A \times B$ and is defined by $A \times B = \{(a, b) : a \in A, b \in B\}$. For example, Let $A = \{1, 2\}$, $B = \{a, b\}$. Then $A \times B = \{(1, a), (1, b), (2, a), (2, b)\}$.

Let A, B, C are non empty sets. Then the Cartesian product of A, B, C is

$A \times B \times C = \{(x, y, z) : x \in A, y \in B, z \in C\}$

Let $A_1, A_2, \ldots, A_n$ be a finite collection of nonempty sets. The Cartesian product of the collection is denoted by $A_1 \times A_2 \times \ldots \times A_n$ and is defined by

$A_1 \times A_2 \times \ldots \times A_n = \{(x_1, x_2, \ldots, x_n) : x_1 \in A_1, x_2 \in A_2, \ldots, x_n \in A_n\}$.

If $A_1 = A_2 = \ldots = A_n = A$, then

$A^n = A \times A \times A \times \ldots \times A = \{(x_1, x_2, \ldots, x_n) : x_1, x_2, \ldots, x_n \in A\}$

$= \{(x_1, x_2, \ldots, x_n) : x_i \in A, i = 1, 2, \ldots, n\}$.

Notes

1. The Cartesian product is not necessarily commutative
   $A \times B \neq B \times A$
2. If $n(A) = p$ and $n(B) = q$, then $n(A \times B) = pq$, where $n(A)$ is the cardinal number of A.
3. If $A = \phi$ or $B = \phi$, then $A \times B = \phi$
4. If either A or B is infinite and other is empty, then $A \times B = \phi$
5. If either A or B is infinite and other is non- empty, then $A \times B$ is infinite

Example 4.34 : Let $P = \{a, b, c\}$ and $Q = \{k, l, m, n\}$. Determine the Cartesian product of P and Q.

Solution : The Cartesian product of P and Q are

$P \times Q = \{(a, k), (a, l), (a, m), (a, n), (b, k), (b, l), (b, m), (b, n), (c, k), (c, l), (c, m), (c, n)\}$

Example 4.35 : Prove that $(A \times B) \cap (P \times Q) = (A \cap P) \times (B \cap Q)$

Solution : Let $(x, y) \in (A \times B) \cap (P \times Q)$

$\Rightarrow (x, y) \in (A \times B)$ and $(x, y) \in (P \times Q)$

$\Rightarrow (x \in A$ and $y \in B)$ and $(x \in P$ and $y \in Q)$

$\Rightarrow (x \in A$ and $x \in P)$ and $(y \in B$ and $y \in Q)$

$\Rightarrow x \in (A \cap P)$ and $y \in (B \cap Q)$

$\Rightarrow (x, y) \in (A \cap P) \times (B \cap Q)$

Therefore, $(A \times B) \cap (P \times Q) \subseteq (A \cap P$

Now, conversely let, $(x, y) \in (A \cap P$

$\Rightarrow (x, y) \in (A \cap P)$ and $(x, y) \in ($

$\Rightarrow (x \in A$ and $x \in P)$ and $(y \in$

$\Rightarrow (x \in A$ and $y \in B)$ and $(x \in$

$\Rightarrow (x, y) \in (A \times B)$ and $(x, y) \in$

$\Rightarrow (x, y) \in (A \times B) \cap (P \times Q)$

Therefore, $(A \cap P) \times (B \cap Q) \subseteq$

From (1) and (2), we get $(A \times$

Example 4.36 : Prove that A

Solution : Let $(x, y) \in A \times$

$\Rightarrow x \in A$ and $y \in (B \cap C$

$\Rightarrow x \in A$ and $(y \in B$ and

$\Rightarrow x \in A$ and $y \in B)$ an

$\Rightarrow (x, y) \in (A \times B)$ and

$\Rightarrow (x, y) \in (A \times B) \cap ($

Therefore, $A \times (B \cap C) \subseteq$

Now, conversely let, $(x,$

$\Rightarrow (x, y) \in (A \times B)$ an

$\Rightarrow (x \in A$ and $y \in B$

$\Rightarrow x \in A$ and$(y \in$

$\Rightarrow x \in A$ and $y \in ($

$\Rightarrow (x, y) \in A \times (B$

Therefore, $(A \times B) \cap$

From (1) and (2), w

Example 4.37 : Pro

Solution : Let $(x,$

$\Rightarrow x \in A$ and y

$\Rightarrow x \in A$ and $($

$\Rightarrow x \in A$ and y

$\Rightarrow (x, y) \in (A$

$\Rightarrow (x, y) \in (A$

Therefore, $A \times ($

Now, conversel

$\Rightarrow (x, y) \in (A$

$\Rightarrow (x \in A$ ar

$\Rightarrow x \in (A \cap P)$ and $y \in (B \cap Q)$

$\Rightarrow (x, y) \in (A \cap P) \times (B \cap Q)$

Therefore, $(A \times B) \cap (P \times Q) \subseteq (A \cap P) \times (B \cap Q)$

Now, conversely let, $(x, y) \in (A \cap P) \times (B \cap Q)$

$\Rightarrow (x, y) \in (A \cap P)$ and $(x, y) \in (B \cap Q)$ ... (1)

$\Rightarrow (x \in A$ and $x \in P)$ and $(y \in B$ and $y \in Q)$

$\Rightarrow (x \in A$ and $y \in B)$ and $(x \in P$ and $y \in Q)$

$\Rightarrow (x, y) \in (A \times B)$ and $(x, y) \in (P \times Q)$

$\Rightarrow (x, y) \in (A \times B) \cap (P \times Q)$

Therefore, $(A \cap P) \times (B \cap Q) \subseteq (A \times B) \cap (P \times Q)$

From (1) and (2), we get $(A \times B) \cap (P \times Q) = (A \cap P) \times (B \cap Q)$ ... (2)

Example 4.36 : Prove that $A \times (B \cap C) = (A \times B) \cap (A \times C)$

Solution : Let $(x, y) \in A \times (B \cap C)$

$\Rightarrow x \in A$ and $y \in (B \cap C)$

$\Rightarrow x \in A$ and $(y \in B$ and $y \in C)$

$\Rightarrow x \in A$ and $y \in B)$ and $(x \in A$ and $y \in C)$

$\Rightarrow (x, y) \in (A \times B)$ and $(x, y) \in (A \times C)$

$\Rightarrow (x, y) \in (A \times B) \cap (A \cap C)$

Therefore, $A \times (B \cap C) \subseteq (A \times B) \cap (A \times C)$ ... (1)

Now, conversely let, $(x, y) \in (A \times B) \cap (A \times C)$

$\Rightarrow (x, y) \in (A \times B)$ and $(x, y) \in (A \times C)$

$\Rightarrow (x \in A$ and $y \in B)$ and $(x \in A$ and $y \in C)$

$\Rightarrow x \in A$ and$(y \in B$ and $y \in C)$

$\Rightarrow x \in A$ and $y \in (B \cap C)$

$\Rightarrow (x, y) \in A \times (B \cap C)$

Therefore, $(A \times B) \cap (A \times C) \subseteq A \times (B \cap C)$ ... (2)

From (1) and (2), we get $A \times (B \cap C) = (A \times B) \cap (A \times C)$

Example 4.37 : Prove that $A \times (B \cup C) = (A \times B) \cup (A \times C)$

Solution : Let $(x, y) \in A \times (B \cup C)$

$\Rightarrow x \in A$ and $y \in (B \cup C)$

$\Rightarrow x \in A$ and $(y \in B$ or $y \in C)$

$\Rightarrow x \in A$ and $y \in B)$ or $(x \in A$ and $y \in C)$

$\Rightarrow (x, y) \in (A \times B)$ or $(x, y) \in (A \times C)$

$\Rightarrow (x, y) \in (A \times B) \cup (A \times C)$ ... (1)

Therefore, $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$

Now, conversely let, $(x, y) \in (A \times B) \cup (A \times C)$

$\Rightarrow (x, y) \in (A \times B)$ or $(x, y) \in (A \times C)$

$\Rightarrow (x \in A$ and $y \in B)$ or $(x \in A$ and $y \in C)$

$\Rightarrow x \in A$ and ($y \in B$ or $y \in C$)

$\Rightarrow x \in A$ and $y \in (B \cup C)$

$\Rightarrow (x, y) \in A \times (B \cup C)$

Therefore, $(A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$

From (1) and (2), we get $A \times (B \cup C) = (A \times B) \cup (A \times C)$

Illustrative Examples

1. Determine the power set P (A) of the set A = {1, 2, 3}

Solution The power set of the set A = {1, 2, 3} is

$P(A) = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}, \{1, 2, 3\}, \phi\}$

2. Determine the cardinalities of the sets

(i) $P = \{n^7 : n$ is a positive integer$\}$

(ii) $Q = \{n^{109} : n$ is a positive integer$\}$

(iii) $P \cup Q$

(iv) $P \cap Q$

Solution (i) The cardinality of the set P is infinite as the number of positive integers are infinite.

(ii) The cardinality of the set Q is infinite as the number of positive integers is infinite.

(iii) Since, P and Q are both infinite so their union is also infinite and hence the cardinality of $P \cup Q$ is infinite.

(iv) The cardinality of $P \cap Q$ is one because for $n = 1$

$1^7 = 1^{109}$

for $n = 2$

$2^7 \neq 2^{109}$ and so on.

So, $P \cap Q$ contains only one element. So, cardinality of $P \cap Q$ is one.

3. Let N be the set of all natural numbers. Let P be all finite subsets of N. Determine the cardinality of the set P.

Solution The number of subsets of any set is 2^n, where n is the number of elements in the set. Hence, the cardinality of the set P is 2^n.

4. Prove that for sets A and B that $A \subseteq B$ if and only if $A \cup B = B$. [KU 2015]

Solution Let, $x \in A \cup B$

$\Rightarrow x \in A$ or $x \in B \Rightarrow x \in B$ [$\because A \subseteq B$]

$\therefore A \cup B \subseteq B$

Again, $y \in B \Rightarrow y \in A \cup B$ [$\because A \cup$

$\therefore B \subseteq A \cup B$

Hence, $A \cup B = B$ **[Proved]**

5. Prove that $A \cap (B \Delta C) = (A$

Solution Now $(A \cap B) \Delta (A \cap C$

$= [(A \cap B) \cap (A \cap C)'] \cup [(A \cap$

$= [(A \cap B) \cap (A' \cup C')] \cup [(A' \cap$

$= [(A \cap B \cap A') \cup (A \cap B \cap C$

$= (A \cap B \cap C') \cup (A \cap B' \cap C$

$= A \cap [(B \cap C') \cup (B' \cap C)]$

$= A \cap (B \Delta C)$

6. If $n(A) = 5$ and $n(B) =$ $n(A \cup B)$.

Solution We know that $n(A$

When $n(A \cap B)$ is maximu

We know that $A \cap B \subseteq B$

Maximum value of the va

Maximum value of $n(A \cup$

Minimum Value of $n(A \cup$

7. For any there sub and $A \cap B = A \cap C$

Solution Let, $x \in B \Rightarrow$

Given that $A \cup B = A \cup$

$\Rightarrow x \in A \cup C$

So either $x \in A$ or $x \in$

If $x \in A$

$\Rightarrow x \in A \cap B$ [as x

Given that $A \cap B = A$

$\Rightarrow x \in A \cap C$

$\Rightarrow x \in C$

Thus for both cases,

Therefore, $B \subseteq C$

Similarly, we can est

Hence $B = C$

Again, $y \in B \Rightarrow y \in A \cup B$ [$\because A \cup B = B$]
$\therefore B \subseteq A \cup B$
Hence, $A \cup B = B$ **[Proved]**

5. Prove that $A \cap (B \Delta C) = (A \cap B) \Delta (A \cap C)$

Solution Now $(A \cap B) \Delta (A \cap C)$
$= [(A \cap B) \cap (A \cap C)'] \cup [(A \cap B)' \cap (A \cap C)]$
$= [(A \cap B) \cap (A' \cup C')] \cup [(A' \cup B') \cap (A \cap C)]$ [By using De Morgan's law]
$= [(A \cap B \cap A') \cup (A \cap B \cap C')] \cup [(A' \cap A \cap C) \cup (B' \cap A \cap C)]$
$= (A \cap B \cap C') \cup (A \cap B' \cap C)$
$= A \cap [(B \cap C') \cup (B' \cap C)]$
$= A \cap (B \Delta C)$

6. If $n(A) = 5$ and $n(B) = 3$ Then find the maximum and minimum value of $n(A \cup B)$. [VU 2018]

Solution We know that $n(A \cup B) = n(A) + n(B) - n(A \cap B)$
$= 5 + 3 - n(A \cap B) = 8 - n(A \cap B)$

When $n(A \cap B)$ is maximum then $n(A \cup B)$ is minimum and vice versa.
We know that $A \cap B \subseteq B$ that is $n(A \cap B) \leq n(B) = 3$
Maximum value of the value of $n(A \cap B)$ is 3 and minimum value of $n(A \cap B)$ is 0.
Maximum value of $n(A \cup B)$ is $8 - 0 = 8$.
Minimum Value of $n(A \cup B)$ is $8 - 3 = 5$.

7. For any there subsets A, B and C of a universal set U. If $A \cup B = A \cup C$ and $A \cap B = A \cap C$. Then show that $B = C$. [BU 2019]

Solution Let, $x \in B \Rightarrow x \in A \cup B$
Given that $A \cup B = A \cup C$
$\Rightarrow x \in A \cup C$
So either $x \in A$ or $x \in C$
If $x \in A$
$\Rightarrow x \in A \cap B$ [as $x \in B$]
Given that $A \cap B = A \cap C$
$\Rightarrow x \in A \cap C$
$\Rightarrow x \in C$
Thus for both cases, $x \in C$
Therefore, $B \subseteq C$
Similarly, we can establish that $C \subseteq B$
Hence $B = C$

8. **If A, B, C are three sets such that**
(i) $A \cup C = B \cup C$ and $A \cap C = B \cap C$ then prove that $A = B$
(ii) $A \cup B = A \cup C$ and $A \cap B = A \cap C$ then prove that $C = B$
(iii) $A \cap C = B \cap C$ and $A \cap C' = B \cap C'$ then prove that $A = B$

Solution (i) Given A, B, C are three sets such that $A \cup C = B \cup C$ and $A \cap C = B \cap C$

Now, $A = A \cap (A \cap C)$ [by absorption law] ...(2)
$= A \cup (A \cap C)$ [by distributive law]
$= A \cup (B \cap C)$ [by (2)]
$= (A \cup B) \cap (A \cup C)$
$= (A \cup B) \cap (B \cup C)$
$= (B \cup A) \cap (B \cup C)$
$= B \cup (A \cap C)$
$= B \cup (B \cap C)$ [by (2)]
$= B$

Hence $A = B$.

(ii) Given A,B,C are three sets such that $A \cup B = A \cup C$ and $A \cap B = A \cap C$ ...(3)

Now $B = B \cup (A \cap B)$
$= B \cup (A \cap C)$
$= (B \cup A) \cap (B \cup C)$
$= (A \cup B) \cap (B \cup C)$
$= (A \cup C) \cap (B \cup C)$
$= (A \cap B) \cup C$
$= (A \cap C) \cup C$
$= C$

Hence $B = C$

(iii) Given A, B, C three sets such that $A \cap C = B \cap C$ and $A \cap C' = B \cap C'$

Now $A = A \cap (C \cup C')$
$= (A \cap C) \cup (A \cap C')$
$= (B \cap C) \cup (B \cap C')$
$= B \cap (C \cup C')$
$= B$

Hence $A = B$

9. **Find the number between 1 and 500 that are divisible by 2, 3 and 5.** [MU 2018]

Solution Let $A = \{1 \le x \le 500 : x \in \mathbb{N}, x \text{ is divisible by } 2\}$ So, $n(A) = 250$
$B = \{1 \le x \le 500 : x \in \mathbb{N}, x \text{ is divisible by } 3\}$ So, $n(B) = 166$

$C = \{1 \le x \le 500 : x \in \mathbb{N}, x \text{ is d}$
So, $n(C) = 100$
Now, $A \cap B \cap C$
$= \{1 \le x \le 500 : x \in \mathbb{N}, x \text{ is divisil}$
$= \{1 \le x \le 500 : x \in \mathbb{N}, x \text{ is div}$
So $n(A \cap B \cap C) = 16$

If A is a fixed subset of $A \cup B = B$, then prove

Solution Given A is a fixed s
Since A is a fixed subset of S
If $A = S$, then $A \cup B = S \cup B$
So equation (i) is not true i
Now if $A = \phi$, then $A \cup B =$
So, equation (i) is true for A
Hence $A = \phi$

Find the cardinality

Solution Since the set {0,
= 4. Similarly cardinality of

Show that for any tw
$A \Delta (A \Delta B) = B$

Solution Here, $A \Delta (A \Delta B)$

Prove that $A \cap (B \Delta$

Solution R.H.S $= (A \cap B) \Delta$
$= [(A \cap B) \cap (A \cap C)'] \cup$
$= [(A \cap B) \cap (A' \cup C')] \cup$
$= [(A \cap B \cap A') \cup (A \cap$
$= (A \cap B \cap C') \cup (A \cap B$
$= A \cap [(B \cap C') \cup (B' \cap$
$= A \cap (B \Delta C) =$ L.H.S.

Prove the following
to disprove it
$A \times (B - C) =$

Solution Let (x, y) be an e
Then $(x, y) \in A \times (B - C) \Leftrightarrow$
$\Leftrightarrow [x \in A$ and $y \in B]$ ar

$C = \{1 \le x \le 500 : x \in N, x \text{ is divisible by } 5\}$
So, $n(C) = 100$
Now, $A \cap B \cap C$
$= \{1 \le x \le 500 : x \in N, x \text{ is divisible by } 2 \text{ and } x \text{ is divisible by } 3 \text{ and } x \text{ is divisible by } 5\}$
$= \{1 \le x \le 500 : x \in N, x \text{ is divisible by } 30\}$
So $n(A \cap B \cap C) = 16$

10. If A is a fixed subset of a set S such that for all non-empty subset B of S, $A \cup B = B$, then prove that $A = \phi$

Solution Given A is a fixed subset of a set S : $A \cup B = B, \forall B \subseteq S$
Since A is a fixed subset of S then either $A = S$ or $A = \phi$ —(i)
If $A = S$, then $A \cup B = S \cup B = S$
So equation (i) is not true if $A = S$.
Now if $A = \phi$, then $A \cup B = \phi \cup B = B$
So equation (i) is true for $A = \phi$
Hence $A = \phi$

11. Find the cardinality of P(P({0, 1})), P(A) being power set of A. [BU 2019]

Solution Since the set {0, 1} has cardinality two. So cardinality of P({0, 1}) is $2^2 = 4$. Similarly cardinality of P(P({0, 1})) is $2^4 = 16$.

12. Show that for any two sets A, B
$$A \Delta (A \Delta B) = B$$

Solution Here, $A \Delta (A \Delta B) = (A \Delta A) \Delta B = \phi \Delta B = B$

13. Prove that $A \cap (B \Delta C) = (A \cap B) \Delta (A \cap C)$ [KU, 2019]

Solution R.H.S $= (A \cap B) \Delta (A \cap C)$
$= [(A \cap B) \cap (A \cap C)'] \cup [(A \cap B)' \cap (A \cap C)]$
$= [(A \cap B) \cap (A' \cup C')] \cup [A' \cup B') \cap (A \cap C]$
$= [(A \cap B \cap A') \cup (A \cap B \cap C')] \cup [(A' \cap A \cap C) \cup (B' \cap A \cap C)]$
$= (A \cap B \cap C') \cup (A \cap B' \cap C)$ $[A \cap B \cap A' = \phi, A' \cap A \cap C = \phi]$
$= A \cap [(B \cap C') \cup (B' \cap C)]$
$= A \cap (B \Delta C) =$ L.H.S. **[Proved]**

14. Prove the following set theoretic statement if it is true or give example to disprove it
$$A \times (B - C) = (A \times B) - (A \times C)$$

Solution Let (x, y) be an element of $A \times (B - C)$
Then $(x, y) \in A \times (B - C) \Leftrightarrow x \in A$ and $y \in (B - C)$
$\Leftrightarrow [x \in A \text{ and } y \in B]$ and $[x \in A \text{ and } y \notin C]$

$\Leftrightarrow (x, y) \in (A \times B)$ and $(x, y) \notin (A \times C)$

$\Leftrightarrow (x, y) \in (A \times B) - (A \times C)$

Hence $A \times (B - C) = (A \times B) - (A \times C)$

15. If $aN = \{ax : x \in N\}$ then find $3N \cap 7N$ where N is the set of natural numbers. [VU 2018]

Solution Here, $3N = \{3x : x \in N\}$ and $7N = \{7x : x \in N\}$

Since gcd (3, 7) = 1

So, there is no common element between 3N and 7N.

16. What do you mean by 'addition modulo n'? Write the elements of Z_5. [KU 2019]

Solution Let $a, b \in Z$ and $n \in N$ then $a +_n b$. The least non-negative reminder where $a + b$ is divided by n, is called 'addition modulo n'.

The elements of Z_5 are {0, 1, 2, 3, 4}.

17. For any three set A,B and C prove that $A \times (B \cup C) = (A \times B) \cup (A \times C)$. [VU 2018]

Solution Let $(x, y) \in A \times (B \cup C)$ that is $x \in A$ and $y \in B \cup C$

i.e. $x \in A$ and $y \in B$ or $y \in C$

i.e. $x \in A$ and $y \in B$ or $x \in A$ and $y \in C$

i.e. $(x, y) \in (A \times B)$ or $(x, y) \in A \times C$

i.e. $(x, y) \in (A \times B) \cup (A \times C)$

$A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$

Let $(x, y) \in (A \times B) \cup (A \times C)$

$(x, y) \in (A \times B)$ or $(x, y) \in A \times C$

$x \in A$ and $y \in B$ or $x \in A$ and $y \in C$

$x \in A$ and $y \in B$ or $y \in C$

$x \in A$ and $y \in B \cup C$

$(x, y) \in A \times (B \cup C)$

$(A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$

$A \times (B \cup C) = (A \times B) \cup (A \times C)$ **[Proved]**

18. Define an enumerable set with an example. Show that the union of a finite set and an enumerable set is enumerable. [KU 2019]

Solution • A set S is said to be enumerable (or denumerable) if S is equipotent with the set N.

For an example of enumerable set, we consider the set of natural number N, the set is enumerable as because of $f: N \to N$ defined by $f(n) = n$, $n \in N$ is a bijection.

• Let M be an enume
with elements. S_1, S
• **Case I** : when M ∩
We define a mappin
$g(n + j)$
So, g is a bijection
Therefore, M ∩ S
• **Case II** : when M
Let $M_1 = M - S$. The
M_1 is an infinite su
By previous case n
Hence, M ∪ S is er

19. Write the law
$A \Delta B = B \Delta A$.

Solution **1st part :**

1. Idempotent Laws
2. Associative Laws
3. Commutative Laws
4. Distributive Laws
5. De Morgan's Laws
6. Identity Laws
7. Complement Law
8. Involution Law

2nd part : Prove b
Let $x \in A \Delta B \Rightarrow x$
By Commutative
$\Rightarrow x \in (B - A)$
Thus, we have A
Similarly, we can
Therefore, $A \Delta B$

- Let M be an enumerable set with elements $m_1, m_2, m_3, \ldots$ and S be a finite set with elements. $S_1, S_2, \ldots, S_n$
- **Case I :** when $M \cap S = \phi$

We define a mapping such that $g: N \to M \cap S$ such that $g(j) = S_j, j = 1, 2, 3, \ldots, n$

$g(n + j) = m_j, j = 1, 2, 3 \ldots$

So, g is a bijection.

Therefore, $M \cap S$ is an enumerable set.

- **Case II :** when $M \cap S = \phi$

Let $M_1 = M - S$. Then $M_1 \cup S = M \cup S$ and $M_1 \cap S = \phi$

M_1 is an infinite subset of the enumerable set M and thus M_1 is enumerable.

By previous case no. I, $M_1 \cup S$ is enumerable.

Hence, $M \cup S$ is enumerable.

19. Write the laws of algebra of sets. Prove by using law of algebra of sets $A \Delta B = B \Delta A$. [KU 2019]

Solution 1st part : The below table represents the laws of algebra of sets :

1. Idempotent Laws	(a) $A \cup A = A$	(b) $A \cap A = A$
2. Associative Laws	(a) $(A \cup B) \cup C = A \cup (B \cup C)$	(b) $(A \cap B) \cap C = A \cap (B \cap C)$
3. Commutative Laws	(a) $A \cup B = B \cup A$	(b) $A \cap B = B \cap A$
4. Distributive Laws	(a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	(b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
5. De Morgan's Laws	(a) $(A \cup B)^c = A^c \cap B^c$	(b) $(A \cap B)^c = A^c \cup B^c$
6. Identity Laws	(a) $A \cup \phi = A$ (b) $A \cup U = U$, where U is the universal set and is the null set.	(c) $A \cap U = A$ (d) $A \cap \phi = \phi$, where U is the universal set and is the null set.
7. Complement Laws	(a) $A \cup A^c = U$ (b) $A \cap A^c = \phi$, where U is the universal set and ϕ is the null set.	(c) $U^c = \phi$ (d) $\phi^c = U$, where U is the universal set and ϕ is the null set.
8. Involution Law	(a) $(A^c)^c = A$	

2nd part : Prove by using law of algebra of sets $A \Delta B = B \Delta A$

Let $x \in A \Delta B \Rightarrow x \in (A - B) \cup (B - A)$

By Commutative Laws of the algebra of sets, we have

$\Rightarrow x \in (B - A) \cup (A - B) \Rightarrow x \in B \Delta A$

Thus, we have $A \Delta B \subseteq B \Delta A$

Similarly, we can show that $B \Delta A \subseteq A \Delta B$

Therefore, $A \Delta B = B \Delta A$ **[Proved]**

20. **If A and B be two subsets of a set X then prove that $A \subset B$ if and only if $X - B \subset X - A$.** [VU 2019]

Solution Let $A \subset B$ and $x \in X - B$ that is $x \in X$ but x does not belong to B.

Since $A \subset B$ so x does not belongs to A.

$x \in X - A$

$X - B \subset X - A$.

Conversely

Let $p \in A$

So p does not belong to $X - A$

So p does not belong to $X - B$

21. **Let $A_n = \{x : x$ is a multiple of $n, n \in \mathbb{N}\}$**

(i) Find $A_2 \cup A_7$

(ii) Find $A_4 \cap A_6$ [VU 2019]

Solution (i) $A_2 = \{x : x$ is a multiple of 2$\}$

$A_7 = \{x : x$ is a multiple of 7$\}$

$A_2 \cup A_7 = \{x : x$ is a multiple of 2 and x is a multiple of 7$\}$

$A_2 \cup A_7 = \{x : x$ is a multiple of 14$\}$

(ii) $A_4 = \{x : x$ is a multiple of 4$\}$, $A_6 = \{x : x$ is a multile of 6$\}$

gcd(4, 6) = 12

$A_4 \cap A_6 = \{x : x$ is a multiple of 12$\}$

22. **For the sets A = {1, 3, 5, 7, 9}, B = {2, 4, 6, 8} and C = {3, 6, 9} Verify distributive law.** [VU 2019]

Solution Distributive law is

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

$B \cup C = \{2, 4, 6, 8, 3, 6, 9\}$. $A \cap (B \cup C) = \{3, 9\}$

$A \cap B = \phi$ and $A \cap C = \{3, 9\}$

$(A \cap B) \cup (A \cap C) = \{3, 9\} = A \cap (B \cup C)$

$B \cap C = \{6\}$ $A \cup (B \cap C) = \{1, 3, 5, 7, 9, 6\}$

$(A \cup B) = \{1, 3, 5, 7, 9, 2, 4, 6, 8\}$

$(A \cup C) = \{1, 3, 5, 7, 9, 6\}$

$(A \cup B) \cap (A \cup C) = \{1, 3, 5, 7, 9, 6\}$

It follows distributive law.

23. **Find how many integers between 1 and 60 that are not divisible by 2 nor by 3 and nor by 5. Also determine the number of integers divisible by 5 not by 2, not by 3 with set theoretic approach.** [VU 2019]

Solution Let $A = \{1 \le x \le 60 : x \in \mathbb{N}, x$ is divisible by 2$\}$ So $n(A) = 30$

$B = \{1 \le x \le 60 : x \in$

$C = \{1 \le x \le 60 : x \in$

$A \cap B = \{1 \le x \le 60 : x$

$= \{1 \le x \le 60 : x$

$B \cap C = \{1 \le x \le 60 : x$

$= \{1 \le x \le 60 :$

$A \cap C = \{1 \le x \le 60 :$

$= \{1 \le x \le 60 :$

$A \cap B \cap C = \{1 \le x \le$

$= \{1 \le x \le$

We know that

$n(A \cup B \cup C) = n(A)$

$= 30 +$

No of integer not d

$= (A \cup B \cup$

$C = \{5, 10, 15, 20,$

$C \cap A = \{10, 20, 30$

$B \cap C = \{15, 30, 4$

$n((C \cap A) \cup (B \cap$

No of integer divi

$= n(C) - n$

24. **Difference**

Solution A set X

element of Y.

A set X is said to

least one element i

Let Y = {1, 2, 3}.

Here X is a prop

and every el

But when X = {

Then X is a sub

exist any element

25. **Let A_0, A_1**

$i = 0, 1, 2$

Solution Here

$B = \{1 \le x \le 60 : x \in N, x \text{ is divisible by } 3\}$ So $n(B) = 20$

$C = \{1 \le x \le 60 : x \in N, x \text{ is divisible by } 5\}$ So $n(C) = 12$

$A \cap B = \{1 \le x \le 60 : x \in N, x \text{ is divisible by 2 and } x \text{ is divisible by } 3\}$

$= \{1 \le x \le 60 : x \in N, x \text{ is divisible by } 6\}$ So $n(A \cap B) = 10$

$B \cap C = \{1 \le x \le 60 : x \in N, x \text{ is divisible by 3 and } x \text{ is divisible by } 5\}$

$= \{1 \le x \le 60 : x \in N, x \text{ is divisible by } 15\}$ So $n(B \cap C) = 4$

$A \cap C = \{1 \le x \le 60 : x \in N, x \text{ is divisible by 2 and } x \text{ is divisible by } 5\}$

$= \{1 \le x \le 60 : x \in N, x \text{ is divisible by } 10\}$ So $n(A \cap C) = 6$

$A \cap B \cap C = \{1 \le x \le 60 : x \in N, x \text{ is divisible by 2 and } x \text{ is divisible by 3 and } x \text{ is divisible by } 5\}$

$= \{1 \le x \le 60 : x \in N, x \text{ is divisible by } 30\}$ So $n(A \cap B \cap C) = 2$

We know that

$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$

$= 30 + 20 + 12 - 10 - 4 - 6 + 2 = 44$

No of integer not divisible by 2 nor by 3 and nor by 5

$= (A \cup B \cup C)^c = 60 - n(A \cup B \cup C) = 60 - 44 = 16$

$C = \{5, 10, 15, 20, 25, 30, 35, 40, 45, 50, 55, 60\}$

$C \cap A = \{10, 20, 30, 40, 50, 60\}$

$B \cap C = \{15, 30, 45, 60\}$

$n((C \cap A) \cup (B \cap C)) = n(C \cap A) + n(B \cap C) - n(A \cap B \cap C) = 6 + 4 - 2 = 8$

No of integer divisible by 5 not by 2, not by 3 $= n(C) - n(C \cap (A \cup B))$

$= n(C) - n((C \cap A) \cup (B \cap C)) = 12 - 8 = 4$

Q **Difference between proper subset and subset with suitable example.** [VU 2018]

Solution A set X is said to be a subset of a set Y if every element of X is an element of Y.

A set X is said to be a proper subset of Y if X is a subset of Y and if there exist at least one element in Y which is not in X.

Let $Y = \{1, 2, 3\}$. $X = \{1, 2\}$

Here X is a proper subset of Y because $3 \in Y$ but $3 \notin X$

and every element of X is an element of Y.

But when $X = \{1, 2, 3\}$

Then X is a subset of Y but it is not proper subset of Y because there does not exist any element $x \in Y$ such that $x \notin X$.

Q **Let A_0, A_1 and A_2 be three subsets of $\mathbb{Z}$ defined by $A_i = \{3n + i : n \in \mathbb{Z}\}$ for $i = 0, 1, 2$. Show that A_0, A_1 and A_2 form a partition of $\mathbb{Z}$.**

Solution Here $A_0 = \{3^n + 0 : n \in \mathbb{Z}\} = \{3n : n \in \mathbb{Z}\}$

$A_1 = \{3n + 1 : n \in \mathbb{Z}\}$
$A_2 = \{3n + 2 : n \in \mathbb{Z}\}$
Here (a) $A_0 \cap A_1 = \phi$. and $A_1 \cap A_2 = \phi$ and $A_0 \cap A_2 = \phi$.
Also $A_0 \cup A_1 \cup A_2 = \{n \in \mathbb{N}$: when n is divisible by 3 or n gives remainder 1 or 2 when n is divisible by

$= \mathbb{Z}$

By the definition of partition of a set we can say that A_0, A_1 and A_2 form a partition of $\mathbb{Z}$.

26. **Prove that $A - B = A \cap B^c$** [VU 2019, 2020]

Solution Let $x \in A - B$. that is $x \in A$ but $x \notin B$
That is $x \in A$ and $x \in B^c$
$x \in A \cap B^c$
So, $A - B \subset A \cap B^c$
Let $x \in A \cap B^c$
That is $x \in A$ and $x \in B^c$ so, $x \in A$ but $x \notin B$
Hence, $x \in A - B$ thus, $A \cap B^c \subset A - B$
Hence, $A - B = A \cap B^c$

27. **Find the following set in Set-builder form $A = \{1, 8, 27, 64 ...\}$.** [VU 2020]

Solution $A = \{n^3 : n$ is a natural number$\}$

28. **Represent the following set-in tabular form $A = \{x : x = 2n\}$, (n being a natural number).** [VU 2020]

Solution The tabular form $A = \{x : x = 2n\}$ (n being a natural number) is
$A = \{2, 4, 6, 8, 10, 12, ...\}$

29. **Find the total number of distinct relations from a finite set A to another finite set B.** [VU 2020]

Solution Let A be a set of m elements, B be a set of n elements.
So, $n(A \times B) = mn$
$n(P(A \times B)) = 2^{mn}$ where $P(A \times B)$ is the power set of $A \times B$.
∴ The total no of distinct relation from a finite set of m elements to a finite set of n elements is 2^{mn}.

30. **Define Universal relation and empty relation** [VU 2020]

Solution **Universal relation :** For a set A, $A \times A$ is called universal relation on A.
Empty Relation : For a set A, ϕ is callled empty relation on A.

31. **If $A = \{x : -1 \le x \le 6\}$. and $B = \{x : x >$**

Solution $A \cup B = \{x : x \ge -1\}$ and $A \cap B =$

32. **Let A, B and C be three sets such that $A \cup C = B \cup C$ and $A \cap C = B \cap C$ then prove that A = B.**

Solution Let $x \in A$ *i.e.* $x \in A \cup C$
i.e., $x \in B \cup C$
$x \in B$ or. $x \in C$
If $x \in B$ then $A \subset B$.
If $x \in C$ then $x \in A \cap C = B \cap C$
i.e., $x \in B$. $A \subset B$
Now Let $x \in B$. *i.e.* $x \in B \cup C$
i.e., $x \in A \cup C$
$x \in A$. or $x \in C$
If $x \in A$ then $B \subset A$
If $x \in C$ then $x \in B \cap C = A \cap C$
i.e., $x \in A$ then $B \subset A$
So, $A = B$.

33. **If a finite set has n elements**

Solution We prove it by mathemat
to set say T.
For the basic step suppose $X = \phi$ t
The empty set has one subset $n(P$
Now for induction hypothesis let
Now for inductive step let T has (
Every subset of X is a subset of T.
Any subset of T either contains t
If a subset of T does not contain
those subsets.
On the other hand if a subset of
including 'a' in one of the 2^k subse
So total no of subsets of T is 2^k
This completes the proof.

34. **If $I = \{1, 2, 3\}$ and $J = \{a, b$**

Solution $I \times J = \{(1, a), (1, b), (2,$
$J \times I = \{(a, 1), (b, 1), (a, 2), (b, 2$

31. If $A = \{x : -1 \le x \le 6\}$, and $B = \{x : x > 3\}$. Find $A \cup B$ and $A \cap B$. [VU 2020]

Solution $A \cup B = \{x : x \ge -1\}$ and $A \cap B = \{x : 3 < x \le 6\}$

32. Let A, B and C be three sets such that $A \cap C = B \cap C$ and $A \cup C = B \cup C$ then prove that $A = B$. [VU 2020]

Solution Let $x \in A$ *i.e.* $x \in A \cup C$
i.e., $x \in B \cup C$
$x \in B$ or $x \in C$
If $x \in B$ then $A \subset B$.
If $x \in C$ then $x \in A \cap C = B \cap C$
i.e., $x \in B$, $A \subset B$
Now Let $x \in B$, *i.e.* $x \in B \cup C$
i.e., $x \in A \cup C$
$x \in A$, or $x \in C$
If $x \in A$ then $B \subset A$
If $x \in C$ then $x \in B \cap C = A \cap C$
i.e., $x \in A$ then $B \subset A$
So, $A = B$.

33. If a finite set has n elements show that the power set has 2^n elements. [VU 2018]

Solution We prove it by mathematical induction on the number of elements of the set say T.
For the basic step suppose $X = \phi$ that is $n(X) = 0$
The empty set has one subset $n(P(X)) = 2^0 = 1$
Now for induction hypothesis let $n(X) = k$ so $n(P(X)) = 2^k$.
Now for inductive step let T has $(k + 1)$ elements. that is $T = X \cup \{a\}$.
Every subset of X is a subset of T.
Any subset of T either contains the element 'a' or it does not contain 'a'.
If a subset of T does not contain 'a' then it is also a subset of X and there are 2^k of those subsets.
On the other hand if a subset of T does contain 'a' then that subset is formed by including 'a' in one of the 2^k subsets of X. So, T has 2^k subsets containing 'a'.
So total no of subsets of T is $2^k + 2^k = 2^{k+1}$.
This completes the proof.

34. If $I = \{1, 2, 3\}$ and $J = \{a, b\}$. List all the members of $A \times B$ and $B \times A$.

Solution $I \times J = \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$
$J \times I = \{(a, 1), (b, 1), (a, 2), (b, 2), (a, 3), (b, 3)\}$

35. **Is the set E = {a : a + 7 = 7} a null set?**

Solution From $a+7=7 \Rightarrow a=0$. Thus the set E has an element zero. So the given set E is not a null set.

36. **If A = {a, b, c, d, e}, B = {c, a, e, g} and C = {b, e, f, g}, then show that $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$**

Solution $A \cup B = \{a, b, c, d, e, g\}$. So, $(A \cup B) \cap C = \{b, e, g\}$

Again, $A \cap C = \{b, e\}$, $B \cap C = \{e, g\}$

Thus, $(A \cap C) \cup (B \cap C) = \{b, e, g\}$.

Hence, $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$

37. **Given that $P \cap Q = P \cap R$, is it necessary that Q = R? Justify the answer.**

Solution This is not necessary

$P \cap Q$ contains elements common to both P and Q.

$P \cap R$ contains elements common to both P and R.

But it is not necessary that Q = R because set R can have elements of set Q which are elements of set P as well, but it can also have elements other than those in set Q.

For e.g., let P = {1, 2, 5, 6}, Q = {1, 2, 8}, R = {1, 2, 9}

Then, $P \cap Q = P \cap R = \{1, 2\}$; but $Q \neq R$

38. **Let A, B, C be arbitrary sets. Prove the following :**

(i) $(A - B) - C = A - (B \cup C)$

(ii) $(A - B) - C = (A - C) - B$

(iii) $(A - B) - C = (A - C) - (B - C)$

Solution (i) Let $x \in (A - B) - C$

$\Rightarrow x \in (A - B), x \notin C$

$\Rightarrow x \in A, x \notin B, x \notin C$

$\Rightarrow x \in A, x \notin B$ or C

$\Rightarrow x \in A, x \notin (B \cup C)$

$\Rightarrow x \in A - (B \cup C)$

Therefore, $(A - B) - C \subseteq A - (B \cup C)$ ...(1)

Conversely, let $x \in A - (B \cup C)$

$\Rightarrow x \in A, x \notin (B \cup C)$

$\Rightarrow x \in A, x \notin B, x \notin C$

$\Rightarrow x \in (A - B), x \notin C$

$\Rightarrow x \in (A - B) - C$

Therefore, $A - (B \cup C) \subseteq (A - B) - C$ ...(2)

From (1) and (2), we get $(A - B) - C = A - (B \cup C)$

(ii) Let $x \in (A - B) - C$

$\Rightarrow x \in (A - B), x \notin C$

$\Rightarrow x \in A, x \notin B, x \notin C$

$\Rightarrow x \in A, x \notin C, x \notin B$

$\Rightarrow x \in (A - C), x \notin B$

$\Rightarrow x \in (A - C) - B$

Therefore, $(A - B) - C \subseteq (A$

Conversely, let $x \in (A - C)$

$\Rightarrow x \in (A - C), x \notin B$

$\Rightarrow x \in A, x \notin C, x \notin B$

$\Rightarrow x \in A, x \notin B, x \notin C$

$\Rightarrow x \in (A - B), x \notin C$

$\Rightarrow x \in (A - B) - C$

Therefore, $(A - C) - B \subseteq (A$

From (1) and (2), we get

(iii) Let $x \in (A - B) - C$

$\Rightarrow x \in (A - B), x \notin C$

$\Rightarrow x \in A, x \notin B, x \notin C$

$\Rightarrow x \in A, x \notin C, x \notin B, x$

$\Rightarrow x \in (A - C), x \notin (B -$

$\Rightarrow x \in (A - C) - (B - C$

Therefore, $(A - B) - C \subseteq$

Conversely, let $x \in (A -$

$\Rightarrow x \in (A - C), x \notin (B -$

$\Rightarrow x \in A, x \notin C, x \notin B,$

$\Rightarrow x \in A, x \notin B, x \notin C$

$\Rightarrow x \in (A - B), x \notin C$

$\Rightarrow x \in (A - B) - C$

Therefore, $(A - C) - (B -$

From (1) and (2), we ge

39. **Using set theory fi**

Solution Let A, B, C be respectively. Then

A = {2, 4, 8, 10, 12, 14, 16,

B = {3, 6, 9, 12, 15, 18, 21,

and C = {4, 8, 12, 16, 20,

$A \cap B \cap C$ = {12, 24, 36,

Now, the least element i

The l.c.m of 2, 3 and 4 is

(ii) Let $x \in (A - B) - C$
$\Rightarrow x \in (A - B), x \notin C$
$\Rightarrow x \in A, x \notin B, x \notin C$
$\Rightarrow x \in A, x \notin C, x \notin B$
$\Rightarrow x \in (A - C), x \notin B$
$\Rightarrow x \in (A - C) - B$
Therefore, $(A - B) - C \subseteq (A - C) - B$.
Conversely, let $x \in (A - C) - B$
$\Rightarrow x \in (A - C), x \notin B$ ... (1)
$\Rightarrow x \in A, x \notin C, x \notin B$
$\Rightarrow x \in A, x \notin B, x \notin C$
$\Rightarrow x \in (A - B), x \notin C$
$\Rightarrow x \in (A - B) - C$
Therefore, $(A - C) - B \subseteq (A - B) - C$.
From (1) and (2), we get $(A - B) - C = (A - C) - B$ ... (2)

(iii) Let $x \in (A - B) - C$
$\Rightarrow x \in (A - B), x \notin C$
$\Rightarrow x \in A, x \notin B, x \notin C$
$\Rightarrow x \in A, x \notin C, x \notin B, x \notin C$
$\Rightarrow x \in (A - C), x \notin (B - C)$
$\Rightarrow x \in (A - C) - (B - C)$
Therefore, $(A - B) - C \subseteq (A - C) - (B - C)$ ... (1)
Conversely, let $x \in (A - C) - (B - C)$
$\Rightarrow x \in (A - C), x \notin (B - C)$
$\Rightarrow x \in A, x \notin C, x \notin B, x \notin C$
$\Rightarrow x \in A, x \notin B, x \notin C$
$\Rightarrow x \in (A - B), x \notin C$
$\Rightarrow x \in (A - B) - C$
Therefore, $(A - C) - (B - C) \subseteq (A - B) - C$. ... (2)
From (1) and (2), we get $(A - B) - C = (A - C) - (B - C)$

39. Using set theory find the least common multiplier (l.c.m) of 2, 3 and 4.

Solution Let A, B, C be the sets containing the multiples of 2, 3 and 4 respectively. Then
$A = \{2, 4, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 30, 32, 34, 36, ...\}$
$B = \{3, 6, 9, 12, 15, 18, 21, 24, 27, ...\}$
and $C = \{4, 8, 12, 16, 20, 24, 28, 32, 36, 40, 44, ...\}$
$A \cap B \cap C = \{12, 24, 36, ...\}$ = set of all common multiples of 2, 3 and 4
Now, the least element in $A \cap B \cap C$ is 12
The l.c.m of 2, 3 and 4 is 12

40. **An inquiry into 1000 candidates who failed in Mathematics at class XII examination revealed the following data :**

658 failed in aggregate
372 failed in paper I
590 failed in paper II
166 failed in aggregate and paper I
434 failed in aggregate and paper II
126 failed in both papers

Find out how many candidates failed in

(i) all the three,
(ii) aggregate but not in paper II,
(iii) paper I but in aggregate,
(iv) paper II but not in paper I,
(v) aggregate or paper II but not in paper I,
(vi) aggregate but not in paper I and paper II.

Solution Let $n(A)$ = number of students failed in aggregate = 658,

$n(B)$ = number of students failed in paper I = 372,

$n(C)$ = number of students failed in paper II = 590,

$n(A \cap B)$ = number of students failed in aggregate and paper I = 166,

$n(B \cap C)$ = number of students failed in paper I and paper II = 126,

$n(C \cap A)$ = number of students failed in aggregate and paper II = 434,

$n(A \cap B \cap C)$ = number of students failed in all three subjects,

$n(A \cup B \cup C)$ = total number of students = 1000.

Now, by Fundamental Principle of counting

(i) We have

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$$

$$\Rightarrow 1000 = 658 + 372 + 590 - 166 - 126 - 434 + n(A \cap B \cap C)$$

Thus, the number of students failed in all three subjects is $n(A \cap B \cap C) =$

(ii) Number of students failed in aggregate but not in paper II

$$= n(A) - n(A \cap C) = 658 - 434 = 224$$

(iii) Number of students failed in paper I but not in aggregate

$$= n(B) - n(A \cap B) = 372 - 166 = 206$$

(iv) Number of students failed in paper II but not in paper I

$$= n(C) - n(B \cap C) = 590 - 126 = 464$$

(v) Number of students failed in aggregate or paper II but not in paper I

$$= n[(A \cup C) - B)] = n(A) + n(C) - n(C \cap A) - n(A \cap B) - n(B \cap C) + n(A \cap B \cap C)$$

$$= 658 + 590 - 434 - 166 - 126 + 106 = 628$$

(vi) Number of students failed in aggregate but not in paper I and paper II

$$= n[A - (B \cup C)] = n(A) - n(A \cap B) - n(C \cap A) + n(A \cap B \cap C)$$

$$= 658 - 166 - 434 + 106 = 164$$

41. **In an examination Mathematics, 27% the subjects. Find t**

Solution Let the sets $n(E$ passed in English and Math

Now, $n(E \cup M) = (100$

By Fundamental Princ

$n(E \cup M) = n(E) + n(M$

$\Rightarrow 73\% = 70\% + 65$

Now, 62% = 248 (give

So, total number of c

42. **In a survey of 1 40 had their ov both the colleg borrowed book How many stu**

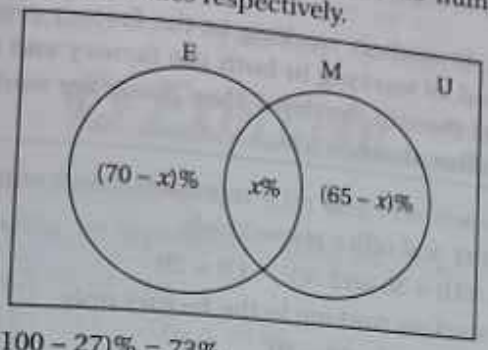
Solution Let $n(A) = n$

$n(B)$ = number of st

$n(C)$ = number of st

41. **In an examination 70% of the candidates passed in English, 65% in Mathematics, 27% failed in both the subjects and 248 passed in both the subjects. Find the total number of candidates.**

Solution Let the sets $n(E)$ and $n(M)$ represent the number of students who passed in English and Mathematics respectively.



Now, $n(E \cup M) = (100 - 27)\% = 73\%$

By Fundamental Principle of counting

$n(E \cup M) = n(E) + n(M) - n(E \cap M)$

$\Rightarrow 73\% = 70\% + 65\% - x\% \Rightarrow x\% = 62\%$

Now, 62% = 248 (given)

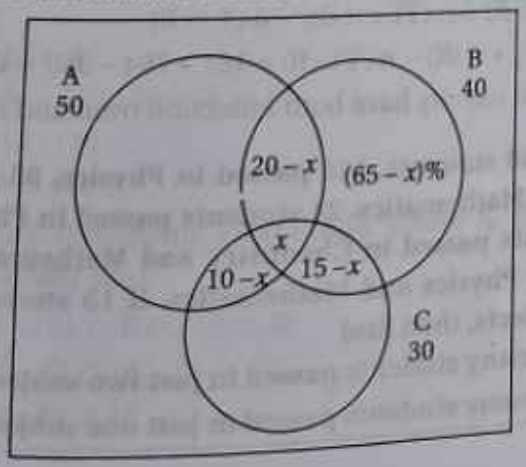
So, total number of candidates $= \frac{248 \times 100}{62} = 400$

42. **In a survey of 100 students it was found that 50 used the college library, 40 had their own library and 30 borrowed books. Out of these 20 used both the college library and their own. 15 used their own library and borrowed books and 10 used the college library and borrowed books. How many students used all the three sources of books.**

Solution Let $n(A)$ = number of students using college library

$n(B)$ = number of students using their own library

$n(C)$ = number of students who borrow books



Suppose x students use all the three sources, then by Fundamental Principle of counting

$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$

$\Rightarrow 100 = 50 + 40 + 30 - 20 - 15 - 10 + x \Rightarrow x = 25$

So, total number of students who use all the three sources of books is 25

43. A firm has 40 workers working in the factory premises, 30 working in its office and 20 working in both the factory and the office. How many workers are there in the firm? How many are working in the (i) factory alone (ii) office alone.

Solution Let the sets $n(F)$ and $n(O)$ represent the number of workers working in the factory and office respectively.

Now, $n(F) = 40$, $n(O) = 30$ and $n(F \cap O) = 20$

The number of workers working in the factory only

$= n(F) - n(F \cap O) = 40 - 20 = 20$

The number of workers working in the office only

$= n(O) - n(F \cap O) = 30 - 20 = 10$

So, the total number of workers $= 20 + 20 + 10 = 50$

44. Making investigations on 1003 families in a city it is observed that 63 families have no induction oven or microwave oven, 794 families have only microwave oven and 187 families have only induction oven. How many families have both induction oven and microwave oven

Solution Let U = set of all 1003 families in the city.

R = set of all families having microwave oven

T = set of all families having induction oven

X = set of families having no induction oven or microwave oven

Then, $n(U) = 1003$, $n(R) = 794$, $n(T) = 187$

Again, $n(T \cup R) = n(U) - n(X) = 1003 - 63 = 940$

We know, $n(T \cup R) = n(T) + n(R) - n(T \cap R)$

$n(T \cap R) = n(T) + n(R) - n(T \cup R) = 187 + 794 - 940 = 41$

So, 41 families in the city have both induction oven and microwave oven

45. Among 150 students, 115 passed in Physics, 63 in Chemistry and 56 passed in Mathematics. 24 students passed in Physics and Chemistry, 25 students passed in Chemistry and Mathematics and 38 students passed in Physics and Mathematics. If 15 students passed in all the three subjects, then find

(i) how many students passed in just two subjects

(ii) how many students passed in just one subject

Solution Let U = set of all 150 stude

P = set of all students who passed

C = set of all students who passed

M = set of students who passed in

$\therefore n(U) = 150$, $n(P) = 115$, $n(C) =$

$n(C \cap M) = 25$, $n(M \cap P) = 38$

Number of students passing only

A Short and Long Answer Type Qu

1. Simplify the following set e

(i) $\overline{(A \cup B)} \cap \overline{(\bar{A} \cup \bar{C})} \cap (\bar{B}$

(ii) $(A \cap B) \cup (A \cap B) \cap (\bar{C}$

(iii) $(A-B) \cup (A \cap B)$

2. Given that U = {1, 2, 3, C = {1, 2, 3, 5, 7} and D =

(i) $(A \cup B) \cap C$

(iii) $\bar{C} \cup \bar{D}$

(v) $A \cup (B - C)$

(vii) $B - (C - D)$

(ix) $(A - B) \cup (C - D)$

3. Prove the following ana

(i) $A - B = A \cap \bar{B}$

(iii) $(A \cap B) \cup (A \cap \bar{B}) =$

(v) $(A \cup B) \cap (A \cup \phi) =$

(vii) $\overline{(A-B)} = \bar{A} \cup B$

(ix) $A - B = \bar{B} - \bar{A}$

(xi) $(A - B) \cap (B - A) = \phi$

(xii) $(A - B) \cup (B - A) = ($

(xiii) $(A \cap B) \subset A \subset (A \cup$

(xiv) $(A \cap B) \subset B \subset (A \cup B$

4. Prove the following s
Verify them graphical

(i) $A \cup B = (A \cap B) \cup$

(ii) $(A \cap B) - C = (A - C$

(iii) $A - (B \cup C) = (A - B$

Solution Let U = set of all 150 students
P = set of all students who passed in Physics
C = set of all students who passed in Chemistry
M = set of students who passed in Mathematics
$\therefore n(U) = 150$, $n(P) = 115$, $n(C) = 63$, $n(M) = 56$, $n(P \cap C) = 24$,
$n(C \cap M) = 25$, $n(M \cap P) = 38$ and $n(P \cap C \cap M) = 15$
Number of students passing only Physics and Chemistry = 24 – 15 = 9

Exercise

Short and Long Answer Type Questions

1. Simplify the following set expressions, using set identities :
   (i) $\overline{(A \cup B)} \cap (\overline{A} \cup \overline{C}) \cap \overline{(\overline{B} \cup C)}$
   (ii) $(A \cap B) \cup (A \cap B) \cap (\overline{C} \cap D)(\overline{A} \cap B)$
   (iii) $(A-B) \cup (A \cap B)$

2. Given that U = {1, 2, 3, 4, ..., 9, 10}, A = {1, 2, 3, 4, 5}, B = {1, 2, 4, 8} C = {1, 2, 3, 5, 7} and D = {2, 4, 6, 8}, find each of the following :
   (i) $(A \cup B) \cap C$
   (ii) $A \cup (B \cap C)$
   (iii) $\overline{C} \cup \overline{D}$
   (iv) $(A \cup B) - C$
   (v) $A \cup (B - C)$
   (vi) $(B - C) - D$
   (vii) $B - (C - D)$
   (viii) $(A \cup B) - (C \cup D)$
   (ix) $(A - B) \cup (C - D)$

3. Prove the following analytically or graphically :
   (i) $A - B = A \cap \overline{B}$
   (ii) $A - (A \cap B) = A - B$
   (iii) $(A \cap B) \cup (A \cap \overline{B}) = A$
   (iv) $A \cup (A \cap B) = A$
   (v) $(A \cup B) \cap (A \cup \phi) = A$
   (vi) $(A \cap B) \cup (B - A) = B$
   (vii) $\overline{(A-B)} = \overline{A} \cup B$
   (viii) $A \cap (B\text{-}A) = \phi$
   (ix) $A - B = \overline{B} - \overline{A}$
   (x) $(A - B) \cup (A \cap B) = A$
   (xi) $(A - B) \cap (B - A) = \phi$
   (xii) $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$
   (xiii) $(A \cap B) \subset A \subset (A \cup B)$
   (xiv) $(A \cap B) \subset B \subset (A \cup B)$.

4. Prove the following statements analytically, where A, B, and C are sets. Verify them graphically also.
   (i) $A \cup B = (A \cap B) \cup (A \cap \overline{B}) \cup (A \cap B)$
   (ii) $(A \cap B) - C = (A - C) \cap (B - C)$
   (iii) $A - (B \cup C) = (A - B) \cap (A - C)$

(iv) $(B \cup C) - A = (B - A) \cup (C - A)$

(v) $(A - B) - C = A - (B \cup C)$

(vi) $(A - B) - C = (A - C) - (B - C)$

(vii) $A \cap (B - C) = (A \cap B) - (A \cap C)$.

5. In a survey of 200 musicians, it was found that 40 wore gloves on the left hand and 39 wore gloves on the right hand. If 160 wore no gloves at all, how many wore a glove on only the right hand? Only the left hand? On both hands?

6. Out of 1200 students in a college, 582 took Economics, 627 took English, 543 took Mathematics. 211 took both Economics and English, 307 took both Economics and Mathematics, 250 took both Mathematics and English, 222 took all the three courses. How many took none of the three?

7. 40 computer programmers interviewed for a job, 25 knew Java, 26 knew Oracle and 7 knew neither Language. How many knew both Languages?

8. Among 100 students, 32 study Mathematics, 20 study Physics, 45 study Biology, 15 study Mathematics and Biology, 7 study Mathematics and Physics, 10 study Physics and Biology and 30 do not study any of the three subjects

(i) Find the number of students studying all three subjects

(ii) Find the number of students studying exactly one of the three subjects

9. A survey was conducted among 100 people. of these 595 like Metro Channel, 595 like Star Movies and 550 like Zee TV; 385 of them like Metro Channel and Star Movies, 350 of them like Metro Channel and Zee TV and 400 of them like Star Movies and Zee TV; 250 of them like Metro Channel, Star Movies and Zee TV. How many of them who do not like Metro channel, do not like Star Movies and do not like Zee TV?

10. Among the first thousand positive integers

(i) Determine the integers which are not divisible by 5, nor by 7, nor by 9

(ii) Determine the integers divisible by 5, but not by 7, not by 9.

11. Determine the Cardinalities of the Set

(i) $P = \{n^7 : n \text{ is a positive integer}\}$

(ii) $Q = \{n^{209} : n \text{ is a positive integer}\}$

(iii) $P \cup Q$

(iv) $P \cap Q$

12. For any three non-null sets A,B,C show that

(i) $[A \cup (B' \cap C')]' = \phi$ if $(B \cup C) \subseteq A$

(ii) $(A - B) \cup B = A$ if $B \subseteq A$

(iii) $[A \cap (B \cup C)] \cap [A' \cup (B' \cap C')] = \phi$

(iv) $(A \cup B) \cap (B \cup C) \cap (C \cup A) = (A \cap B) \cup (B \cap C) \cup (C \cap A)$

(v) $A \cap (B - C) = (A \cap B) \cap$

(vi) $A \cap (B - C) = (A \cap B) -$

(vii) $A \cup (B - C) = (A \cup B) -$

(viii) $A - (B - C) = (A - B) \cup$

(ix) $A - (B \cup C) = (A - B) \cap$

(x) $A - (B \cap C) = (A - B) \cup$

(xi) $A \times (B \cup C) = (A \times B) \cup$

(xii) $A \times (B \cap C) = (A \times B) \cap$

Multiple Choice Questions (M

1. Let A and B be any two equal to

(a) $A \cup B$ (b) A

2. The set $\{x : x \text{ is a vowel}$

(a) $\{a, I, s\}$ (b)

3. Let P(S) denote the po

(a) $P(P(S)) = P(S)$

(c) $P(S) \cap S = P(S)Z$ de

4. The set $\left\{1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}\right.$

(a) $\left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ (b)

5. If P, Q, R are subsets

$(P \cap Q \cap R) \cup (P' \cap$

(a) $Q' \cup R'$

(c) $P' \cup Q' \cup R'$

6. Which of the follow

(a) $\{n \in \mathbb{N}, n \text{ is even}$

(c) $\{x : x \in \mathbb{N} \text{ and } 3x$

7. For a set A the pow of the following op

(a) $\phi \in 2^A$

(c) $\{5, \{6\}\} \in 2^A$

8. Which of the follo

(a) $\{x : (2x - 1)(2x -$

(b) $\{x : x^2 - 4 = 0$ ar

(c) $\{x : 1 < x < 1.1\}$

(d) $\{x : \cos x = 0\}$

(vi) $A \cap (B - C) = (A \cap B) \cap (A \cap C')$
(vii) $A \cap (B - C) = (A \cap B) - (A \cap C)$
(viii) $A \cup (B - C) = (A \cup B) - (A \cup C)$
(ix) $A - (B - C) = (A - B) \cup (A - C)$
(x) $A - (B \cup C) = (A - B) \cap (A - C)$
(xi) $A - (B \cap C) = (A - B) \cup (A - C)$
(xii) $A \times (B \cup C) = (A \times B) \cup (A \times C)$
(xiii) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

Multiple Choice Questions (MCQ)

1. Let A and B be any two sets. Then the set $(A - B) \cup (B - A) \cup (A \cap B)$ is equal to
(a) $A \cup B$ (b) $A^c \cup B^c$ (c) $A^c \cap B^c$ (d) $A \cap B$

2. The set $\{x : x$ is a vowel of the word 'mathematics'$\}$ =
(a) $\{a, I, s\}$ (b) $\{a, i\}$ (c) $\{a, e, i\}$ (d) none of these

3. Let P(S) denote the power of the set Which of the following is always true?
(a) $P(P(S)) = P(S)$ (b) $P(S) \cap P(P(S)) = \{\phi\}$
(c) $P(S) \cap S = P(S)Z$ defined by a R (d) $S \notin P(S)$

4. The set $\left\{1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \ldots\right\}$ =
(a) $\left\{\frac{1}{n} : n \in N\right\}$ (b) $\left\{\frac{1}{n^2} : n \in N\right\}$ (c) $\left\{n^2 : n \in N\right\}$ (d) $\left\{n^3 : n \in N\right\}$

5. If P, Q, R are subsets of the universal set S, then
$(P \cap Q \cap R) \cup (P' \cap Q \cap R) \cup Q' \cup R'$ is
(a) $Q' \cup R'$ (b) $P \cup Q' \cup R'$
(c) $P' \cup Q' \cup R'$ (d) S

6. Which of the following set is finite
(a) $\{n \in N, n$ is even (b) $\{x : x$ is prime$\}$
(c) $\{x : x \in N$ and $3x - 2 = 0\}$ (d) $\{x : \sin x = 0\}$

7. For a set A the power set of A is denoted by 2^A. If $A = \{5, \{6\}, \{7\}\}$ which one of the following operations is true?
(a) $\phi \in 2^A$ (b) $\phi \subseteq 2^A$
(c) $\{5, \{6\}\} \in 2^A$ (d) $\{5, \{6\}\} \subseteq 2^A$

8. Which of the following set is an empty set
(a) $\{x : (2x - 1)(2x - 3)(2x - 5) = 0$ and $x \in N$
(b) $\{x : x^2 - 4 = 0$ and $x \in R\}$
(c) $\{x : 1 < x < 1.1\}$
(d) $\{x : \cos x = 0\}$

9. Let S be non-empty set and P(S) be the power set of S. If A and B are any two sets, then which one of the following is true?
(a) $P(A \cup B) = P(A) \cup P(B)$ (b) $P(A \cap B) = P(A) \cap P(B)$
(c) $P(A \cup B) = P(A) \cap P(B)$ (d) $P(A \cap B) = P(A) \cup P(B)$

10. Every subset of a finite set is finite
(a) True (b) False

11. The number of non-empty subsets of a set consisting of 8 element is
(a) 256 (b) 255 (c) 128 (d) 127

12. Every super set of an infinite set may not be infinite
(a) True (b) False

13. If $A = \{1, 2, 5, 6\}$ and $B = \{1, 2, 3\}$, then $(A \times B) \cap (B \times A)$ is
(a) $\{(1, 1), (1, 2), (2, 1), (2, 2)\}$ (b) $\{(1, 1), (2, 2), (5, 1), (1, 6)\}$
(c) $\{(1, 1), (2, 1), (6, 1), (3, 2)\}$ (d) $\{(2, 3), (3, 1), (3, 2), (5, 3)\}$

14. A finite set can have infinitely many subsets
(a) True (b) False

15. For any three subsets A,B and C which one of the following statements is correct?
(a) $A \cap B = \phi$ implies $A = \phi$ or $B = \phi$
(b) $A - B = \phi$ implies that $A \subseteq B$
(c) $A \cup B = \phi$ implies that $A \subseteq B$
(d) $A \cap B \neq \phi$, $A \cap C \neq \phi$ implies that $A \cap B \cap C \neq \phi$

16. Let $S = \{\phi, \{\phi\}, a, \{a, \phi\}, b\}$ which of the following is true?
(a) $\{\phi\} \subset S$ (b) $\{a, b\} \not\subset S$
(c) $\{b, \{a\}\} \subset S$ (d) S is an infinite set

17. Let A be a finite set of size n. The no of elements in the power set of $A \times A$ is
(a) 2^{2^n} (b) 2^{n^2} (c) $2n$ (d) 2^n

18. The power sets A_1 and A_2 are equal. Then
(a) $A_1 \subset A_2$ (b) $A_1 = A_2$ (c) $A_1 - A_2 \neq \phi$ (d) $A_2 \subset A_1$

19. The number of elements of the power set P(S) of the set $S = \{\phi, 1, \{2, 3\}\}$ is
(a) 2 (b) 4 (c) 8 (d) None

20. For any two sets A and B, $(A - B) \cap B =$
(a) A (b) $B - A$ (c) ϕ (d) B

21. Out of 500 car owners investigated, 400 owned car A and 200 owned car B, 50 owned both A and B cars. This statement is
(a) Correct (b) Wrong

22. $(A \cap B') \cup (B \cap C) =$
(a) $A' \cup B$ (b) $A' \cup B \cup C$ (c) $A' \cup C'$ (d) $A' \cap B$

23. Power set of empty set has
(a) One
(c) Zero

24. What is the cardinality of
(a) 10 (b) 5

25. Which of the following tw
(a) $A = \{1, 2\}$ and $B = \{1\}$
(b) $A = \{1,2\}$ and $B = \{1, 2,$
(c) $A = \{1, 2, 3\}$ and $B = \{2,$
(d) $A = \{1, 2, 4\}$ and $B = \{1,$

26. What is the cardinality of
(a) 8 (b) 6

27. The members of the set S is ——
(a) {0, 2, 4, 5, 9, 58, 49, 56,
(b) {0, 1, 4, 9, 16, 25, 36, 4
(c) {1, 4, 9, 16, 25, 36, 64, 8
(d) {0, 1, 4, 9, 16, 25, 36, 4

If A and B are sets and A
(a) $A = \phi$
(c) $A = B$

Let S be an infinite set an
$S_1 \cup S_2 \cup S_3 \cup \ldots \cup S$
(a) at least one of the sets
(b) not more than one of t
(c) at least one of the sets
(d) none of these

If X and Y are two sets, the
(a) X
(c) ϕ

If A, B and C are any three
(a) $(A - B) \cup (A - C)$
(c) $(A - B) \cap C$

If 63% of persons like bar
thout the percentage of pe
(a) 40 (b) 39

23. Power set of empty set has exactly ——— subset.
(a) One (b) Two
(c) Zero (d) Three

24. What is the cardinality of the set of odd positive integers less than 10?
(a) 10 (b) 5 (c) 3 (d) 20

25. Which of the following two sets are equal?
(a) A = {1, 2} and B = {1}
(b) A = {1,2} and B = {1, 2, 3}
(c) A = {1, 2, 3} and B = {2, 1, 3}
(d) A = {1, 2, 4} and B = {1, 2, 3}

26. What is the cardinality of the Power set of the set {0, 1, 2}?
(a) 8 (b) 6 (c) 7 (d) 9

27. The members of the set S = $\{x : x$ is the square of an integer and $x < 100\}$ is ———
(a) {0, 2, 4, 5, 9, 58, 49, 56, 99, 12}
(b) {0, 1, 4, 9, 16, 25, 36, 49, 64, 81}
(c) {1, 4, 9, 16, 25, 36, 64, 81, 85, 99}
(d) {0, 1, 4, 9, 16, 25, 36, 49, 64, 121}

28. If A and B are sets and $A \cup B = A \cap B$, then
(a) $A = \phi$ (b) $B = \phi$
(c) A = B (d) none of these

29. Let S be an infinite set and $S_1, S_2, S_3, ..., S_n$ be sets such that $S_1 \cup S_2 \cup S_3 \cup ... \cup S_n = S$, then
(a) at least one of the sets S is a finite set
(b) not more than one of the set S can be inite
(c) at least one of the sets S is an initite set
(d) none of these

30. If X and Y are two sets, then $X \cap (Y \cup X)^C$ equals
(a) X (b) Y
(c) ϕ (d) none of these

31. If A, B and C are any three sets, then $A - (B \cup C)$ is equal to
(a) $(A - B) \cup (A - C)$ (b) $(A - B) \cup C$
(c) $(A - B) \cap C$ (d) $(A - B) \cap (A - C)$

32. If 63% of persons like banana, where 76% like apple. What can be said about the percentage of persons who like both banana and apples
(a) 40 (b) 39 (c) 27 (d) 24

Answers

A. Short and Long Answer Type Questions

1. (i) ϕ (ii) B (iii) A.
2. (i) {1, 2, 3, 5} (ii) {1, 3, 4, 5, 6, 7, 8, 9, 10} (iii) {9, 10} (iv) {4, 8}
   (v) {1, 2, 3, 4, 5, 8} (vi) ϕ (vii) {2, 4, 8} (viii) {1, 3, 4, 5, 8} (ix) {1, 3, 5, 7}
5. Musicians who wore gloves on right hand = 1, Musicians who wore glov
   left hand = 0, Musicians who wore gloves on both hands = 39
6. Students who took none of the three subjects = 0
7. Computer programmer who knew both languages = 20
8. (i) 5 (ii) 45
9. 155
10. (i) 134 (ii) 232
11. (i) Infinite (ii) Infinite (iii) Infinite (iv) 1

B. Multiple Choice Questions (MCQ)

1. a	2. c	3. b	4. b	5. d	6. c	7. b	8. a	9. a
11. b	12. b	13. a	14. b	15. d	16. a	17. a	18. b	19. c
21. b	22. b	23. a	24. b	25. c	26. a	27. b	28. c	29. c
31. d	32. b							

05

Introduction

Relationships be
deal with the relatio
come across many
and daughter and
functions, numbers
different sets. The
is to use ordered p
ordered pairs are c
different relations

5.1 Relati

Let S and T be
subset of S × T fro
s of the set S is sai
i.e., $s \rho t$ holds and
we say $\rho \subseteq S \times S$ is
If (s, t) ρ-, then

Example 5.1 : L
Let, $x \in S$ and
Then $\rho = \{(1, 5)$

Note
Let S be a non e
product S × S.

05

Relation

Introduction

Relationships between elements of sets occur in many ways. In daily life we deal with the relationships such as those in education, business, relatives etc. We come across many relationships such as relation between mother and son, father and daughter and so on. In mathematics we study the relationships between functions, numbers, equations etc. Relation is the link between the members of different sets. The most direct way to express the relationship between two sets is to use ordered pairs made up of elements from each set. Therefore, the set of ordered pairs are called binary relations. This chapter will give a deep insight of different relations and its properties with examples.

5.1 Relation (or Binary Relation)

Let S and T be two non-empty sets. A binary relation ρ between S and T is a subset of S × T from a set S to T. If the ordered pair $(s, t) \in \rho$, then the element s of the set S is said to be related to the element t of the set T by the relation ρ i.e. $s \rho t$ holds and we read it as "s is related to t by ρ". If S and T are equal, then we say $\rho \subseteq S \times S$ is a relation on S.

If $(s, t) \rho$-, then s is said to be not related to t by the relation ρ.

Example 5.1 : Let S = {1, 2, 3, 4} and T = {5, 6, 7}.

Let, $x \in$ S and $y \in$ T and $x \rho y$ holds if x / y.

Then ρ = {(1, 5), (1, 6), (1, 7), (2, 6), (3, 6)}, ρ is a proper subset of S × T.

Note

Let S be a non empty set, A binary relation ρ on S is a subset of Cartesian product S × S.

5.2 Domain and Range of a Relation

The domain of a relation ρ is the set of elements in S which are related to some element in T or it is the set of all first entries of the ordered pairs in ρ. It is denoted by DOM (ρ).

The range of a relation ρ is the set of elements in T which are related to some element in S or it is the set of all second entries of the ordered pairs in ρ. It is denoted by RAN (ρ).

Example 5.2 : Let S = {1, 2, 3, 4} and T = {a, b, c, d} and
ρ = {(1, a), (1, b), (1, c), (2, b), (2, c), (2, d)}, then
DOM (ρ) = {1, 2}
RAN (ρ) = {a, b, c, d}

Example 5.3 : Let S = {1, 2, 3, 4, 5} and ρ = {(1, 1), (2, 3), (3, 4), (3, 5), (4, 5)}.
Then DOM (ρ) = {1, 2, 3, 4}, RAN (ρ) = {1, 3, 4, 5}.

5.3 Complement of a Relation

Consider a relation ρ from a set S to T. The complement of relation ρ denoted by $\bar{\rho}$ is a relation from S to T such that $\bar{\rho}$ = {(a, b): (a, b) $\notin \rho$}

Example 5.4 : Let S = {1, 2, 3} and T = {8, 9} and ρ = {(1, 8), (2, 8), (1, 9), (3, 9)}, then the Cartesian product of S and T is
S × T = {(1, 8), (2, 8), (3, 8), (1, 9), (2, 9), (3, 9)}.

The complement relation $\bar{\rho}$ with respect to S × T is $\bar{\rho}$ = {(3, 8), (2, 9)}

5.4 Inverse of a Relation

Consider a relation ρ from a set S to T. The inverse of relation ρ denoted by ρ^{-1} is a relation from T to S such that $b\,\rho^{-1}a$ iff $a\,\rho\,b$ i.e., ρ^{-1} = {(b, a): (a, b) $\in \rho$}

Example 5.5 : If ρ is a relation on the set of positive integers and ρ is defined as
ρ = {(1, 1), (1, 2), (1, 3), (2, 1), (3, 1), (3, 2), (2, 3)}, then the inverse relation is
ρ^{-1} = {(1, 1), (2, 1), (3, 1), (1, 2), (1, 3), (2, 3), (3, 2)}

Example 5.6 : Inverse of the relation ρ when
S = {1, 2, 3, 4, 5} and ρ = {(1, 1), (2, 3), (3, 4), (3, 5), (4, 5)} is
ρ^{-1} = {(1, 1), (3, 2), (4, 3), (5, 3), (5, 4)}.
DOM (ρ^{-1}) = RAN (ρ); RAN (ρ^{-1}) = DOM (ρ).

5.5 Properties of Relation

The nature or type of a relation can be determined by the property of a relation. There are many properties of relations *i.e.*, reflexive, symmetric, asymmetric, transitive, antisymmetric etc. We shall discuss them in the following sections.

5.5.1 Reflexive Relation

Let S be a non- empty set and ρ be a binary relation on S. The relation ρ is said to be reflexive if $(a, a) \in \rho \; \forall \; a \in S$ i.e., $a \rho a$ holds $\forall \; a \in S$.

5.5.2 Irreflexive Relation

Let S be a non-empty set and ρ be a binary relation on S. The relation ρ is said to be irreflexive if $(a, a) \notin \rho$, $\forall a \in S$ *i.e.*, $a \rho a$ does not holds $\forall a \in S$.

Example 5.7 : Let $S = \{1, 2\}$ and $\rho = \{(1, 1), (2, 2), (1, 2)\}$. This relation is reflexive as for every $a \in S$, $(a, a) \in \rho$, *i.e.*, $(1, 1), (2, 2) \in \rho$. This relation is not irreflexive.

Example 5.8 : Let $S = \{1, 2, 3\}$ and $\rho = \{(1, 2), (2, 1), (1, 3), (2, 3)\}$. This relation is not reflexive as for every $a \in S$, $(a, a) \notin \rho$, *i.e.*, $(1, 1), (2, 2), (3, 3) \notin \rho$. So, this relation is irreflexive.

Example 5.9 : Let $S = \{1, 2, 3\}$ and $\rho = \{(1, 1), (1, 2), (1, 3), (2, 3), (3, 3)\}$. This relation is not reflexive as for every $a \in S$, $(a, a) \notin \rho$, *i.e.*, $(1, 1), (3, 3) \in \rho$ but $(2, 2) \notin \rho$. This relation is also not irreflexive since $\forall \; a \in S$, $(a, a) \notin \rho$, is not satisfied. Here, $(1, 1), (3, 3) \in \rho$.

5.5.3 Symmetric Relation

Consider a binary relation ρ on a set S. The relation ρ is said to be symmetric if for any two elements a, b in S

$(a, b) \in \rho \Rightarrow (b, a) \in \rho$.

5.5.4 Asymmetric Relation

Consider a binary relation ρ on a set S. The relation ρ is said to be a symmetric if for any two elements a, b in S

$(a, b) \in \rho \Rightarrow (b, a) \notin \rho$.

Example 5.10 : Let $S = \{1, 2, 3\}$ and $\rho = \{(1, 2), (2, 1), (3, 2), (2, 3)\}$. This relation is symmetric as for every $(a, b) \in \rho$, we have $(b, a) \in \rho$ i.e., for $(1, 2) \in \rho$, we have $(2, 1) \in \rho$. Also for $(3, 2) \in \rho$, we have $(2, 3) \in \rho$. But this relation is not asymmetric.

Example 5.11 : Let $S = \{5, 6, 7\}$ and $\rho = \{(5, 5), (5, 6), (6, 7), (7, 7)\}$. This relation is not symmetric as for every $(a, b) \in \rho$, we have $(b, a) \notin \rho$ i.e., for $(5, 6) \in \rho$, $(6, 5) \notin \rho$. This relation is asymmetric as for every $(a, b) \in \rho$, we have $(b, a) \notin \rho$.

Example 5.12 : Let $S = \{5, 6, 7\}$ and $\rho = \{(5, 6), (6, 5), (6, 7), (7, 7)\}$. This relation is not symmetric as for every $(a, b) \in \rho$, we have $(b, a) \notin \rho$. This relation is not asymmetric as for some $(a, b) \in \rho$, we have $(b, a) \in \rho$.

Example 5.13 : Let $S = \{a, b, c, d\}$ and $\rho = \{(a, b), (b, c), (a, d), (a, a), (b, b), (b, a)\}$. a, b are related to itself but c and d are not related to itself, so ρ is not reflexive. Also ρ is not symmetric since $(b, c) \in \rho$ but $(c, b) \notin \rho$.

5.5.5 Transitive Relation

Let S be a non-empty set and ρ be a binary relation on S. The relation ρ is said to be transitive if for any three elements a, b, c in S,

$(a, b) \in \rho$ and $(b, c) \in \rho \Rightarrow (a, c) \in \rho$.

Example 5.14 : Let S = {1, 2, 3} and ρ = {(1, 2), (2, 1), (1, 1), (2, 2)}. This relation is transitive as for every $(a, b), (b, c) \in \rho$, we have $(a, c) \in \rho$ i.e., for (1, 2), (2, 1) $\in \rho$, we have (1, 1) $\in \rho$.

Example 5.15 : Let S = {1, 2, 3} and ρ = {(1, 2), (2, 1), (1, 1), (2, 3), (3, 2)}. This relation is not transitive as for every $(a, b), (b, c) \in \rho$, we have $(a, c) \notin \rho$ i.e., for (2, 3), (3, 2) $\in \rho$, but (2, 2) $\notin \rho$.

5.5.6 Antisymmetric Relation

Consider a binary relation ρ on a set S. The relation ρ is said to be antisymmetric if $(a, b) \in \rho$ implies that $(b, a) \notin \rho$ unless $a = b$. In other words, we can say that if (a, b) and (b, a) belongs to ρ, then $a = b$.

Example 5.16 : Let S = {1, 2, 3} and ρ = {(1, 2), (2, 1), (1, 3), (3, 2)}. This relation is not antisymmetric as $1 \neq 2$ but (1, 2) and (2, 1) both belong to ρ.

Example 5.17 : Let S = {1, 2, 3} and ρ = {(1, 1), (2, 2)}. This relation is Antisymmetric as $a = b$ when both (a, b) and (b, a) both belongs to ρ.

5.6 Equivalence Relation

Let S be a non–empty set and ρ be a binary relation on S. The relation ρ on S is said to be equivalence relation on S, if ρ is reflexive, symmetric and transitive.

Example 5.18 : Let S = {1, 2, 3, 4} and

ρ = {(1, 1), (1, 3), (2, 2), (2, 4), (3, 1), (3, 3), (4, 2), (4, 4)}. This relation is equivalence relation, since it is reflexive, symmetric as well as transitive.

It is Reflexive since, for 1, 2, 3, 4 in S, we have (1, 1), (2, 2), (3, 3) and (4, 4) $\in \rho$

It is Symmetric since, for every (a, b) in S we have (b, a) belongs to S. For e.g. (1, 3) $\in \rho$ we have (3, 1) $\in \rho$.

It is transitive since for any three elements a, b, c in S,

$(a, b) \in \rho$ and $(b, c) \in \rho \Rightarrow (a, c) \in \rho$.

For e.g., (2, 4) $\in \rho$ and (4, 2) $\in \rho \Rightarrow$ (2, 2) $\in \rho$

Example 5.19 : Let ρ be a relation defined on $\mathbb{Z}$ as follows:

$x \rho y$ holds if f $x - y$ is divisible by 5 and $x, y \in \mathbb{Z}$

Since $\forall\, x \in \mathbb{Z}$, $x - x = 0$ is divisible by 5,

$\therefore\ x \rho x$ holds $\forall\, x \in \mathbb{Z}$.

$\therefore\ \rho$ is reflexive.

Let $x, y \in \mathbb{Z}$ such that $x \rho y$ holds

$x \rho y \Rightarrow x - y$ is divisible by 5

$\Rightarrow -(x - y)$ is divisible by 5

$\Rightarrow y - x$ is divisible by 5

$\Rightarrow y \rho x$ holds.

$\therefore\ \rho$ is symmetric

Let, $x, y, z \in \mathbb{Z}$ such that $x \rho y$ a

$\therefore\ x \rho y$ and $y \rho z$

$\Rightarrow x - y$ is divisible by 5 an

$\Rightarrow (x - y + y - z)$ is divisible

$\Rightarrow (x - z)$ is divisible by 5

$\Rightarrow x \rho z$ holds.

$\therefore\ \rho$ is transitive.

$\therefore\ \rho$ is reflexive, symmetric

5.7 Partial Order

Let S be a non–empty se

S is said to be partial orde

transitive.

Example 5.20 : Let us c

set of positive integers.

ρ = {(2, 1), (3, 1), (3, 2),

This relation is a part

transitive because :

It is Reflexive as for ev

It is antisymmetric as

It is transitive as whe

(4, 2) $\in \rho$ and (2, 1) $\in$

Example 5.21 : The re

is a partial order rela

It is reflexive as for e

It is antisymmetric

a iff $a = b$.

It is transitive as fo

a divides c.

5.7.1 Partial Or

The set S together w

ordered set or PO set. W

a and b are related as

Let $x, y \in \mathbb{Z}$ such that $x \rho y$ holds.
$x \rho y \Rightarrow x - y$ is divisible by 5
$\Rightarrow -(x - y)$ is divisible by 5
$\Rightarrow y - x$ is divisible by 5
$\Rightarrow y \rho x$ holds.
$\therefore \rho$ is symmetric
Let, $x, y, z \in \mathbb{Z}$ such that $x \rho y$ and $y \rho z$ both hold
$\therefore x \rho y$ and $y \rho z$
$\Rightarrow x - y$ is divisible by 5 and $y - z$ is divisible by 5
$\Rightarrow (x - y + y - z)$ is divisible by 5
$\Rightarrow (x - z)$ is divisible by 5
$\Rightarrow x \rho z$ holds.
$\therefore \rho$ is transitive.
$\therefore \rho$ is reflexive, symmetric and transitive, so ρ is an equivalence relation on $\mathbb{Z}$

5.7 Partial Order Relation

Let S be a non–empty set and ρ be a binary relation on S. The relation ρ on S is said to be partial order relation on S, if P is reflexive, antisymmetric and transitive.

Example 5.20 : Let us consider the relation $(x, y) \in \rho$, if $x \geq y$ defined on the set of positive integers. Suppose S = {1, 2, 3, 4} and
$\rho = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3), (1, 1), (2, 2), (3, 3), (4, 4)\}$.
This relation is a partial order relation as it is reflexive, antisymmetric and transitive because :
It is Reflexive as for every $a \in S$, $(a, a) \in \rho$ i.e., (1, 1), (2, 2), (3, 3) and $(4, 4) \in \rho$.
It is antisymmetric as whenever (a, b) and $(b, a) \in \rho$, we have $a = b$.
It is transitive as whenever (a, b) and $(b, c) \in \rho$, we have $(a, c) \in \rho$. For e.g., $(4, 2) \in \rho$ and $(2, 1) \in \rho$, implies $(4, 1) \in \rho$.

Example 5.21 : The relation 'divides' defined on the set of natural numbers N is a partial order relation because :
It is reflexive as for every $a \in \mathbb{N}$, a divides a.
It is antisymmetric as for $a, b \in \mathbb{N}$, such that a divides b. It implies b divides a iff $a = b$.
It is transitive as for $a, b, c \in \mathbb{N}$, such that a divides b and b divides c implies a divides c.

5.7.1 Partial Order Set

The set S together with a partial order relation ρ defined on it is called partial ordered set or PO set. We write (S, ρ) is a PO set. In a PO set (S, ρ) if two elements a and b are related as $a \rho b$, we say 'a precedes b' and 'b succeeds a'.

Example 5.22 : Let S be a set and P(S) be its power set, *i.e.*, P(S) is the collection of all subsets of S. Then the relation $\subseteq$ (subset) is a partial order relation on P(S) because :

It is reflexive as $A \subseteq A$ for all sets A in P(S)

It is antisymmetric as $A \subseteq B$ and $B \subseteq A$ imply $A = B$

It is transitive as $A \subseteq B$ and $B \subseteq C$ imply $A \subseteq C$

Example 5.23 : The set of positive integers under the partial order relation '<' is not a PO set as reflexivity does not hold.

Example 5.24 : If ρ is a partial order relation on a set S and ρ^{-1} be the inverse of ρ, then ρ^{-1} is also a partial order relation because :

It is reflexive as $a \rho^{-1} a$ holds $\forall a \in S$

It is antisymmetric as ρ is a partial order relation implies $a = b$

It is transitive as if $a \rho^{-1} b$ and $b \rho^{-1} c$

$\Rightarrow b \rho a$ and $c \rho b$

$\Rightarrow c \rho b$ and $b \rho a$

$\Rightarrow c \rho a$ as ρ is transitive

$\Rightarrow a \rho^{-1} c$

$\Rightarrow \rho^{-1}$ is transitive

Notes

1. If (S, ρ) is a PO set i.e., the relation ρ is a partial order relation on the set S, then any two elements a and b of the set S are said to be comparable if $a \rho b$ or $b \rho a$ i.e., either $(a, b) \in \rho$ or $(b, a) \in \rho$.
2. If in a PO set, for any two elements $a, b \in S$, $a \rho b$ holds, then we say 'a precedes b' and 'b succeeds a'.
3. In a PO set, for any two elements $a, b \in S$, a is called immediate predecessor of b or b is called immediate successor of a, if a precedes b and S contains no element which lies between a and b. For e.g., if $(S, /)$ is a PO set and $S = \{2, 4, 6, 8\}$. Here, divides i.e., '/' is a partial order relation in S which means b is divisible by a. We find that 2 is not a immediate predecessor of 8 since 4 is in between 2 and 8. Here, 4 divides 8 and there is no number between 4 and 8. So, 4 is the immediate predecessor of 8 or 8 is the immediate successor of 4. Similarly, 2 is the immediate predecessor of 4 and 6.

5.7.2 Linearly Ordered or totally ordered Set

A PO set (S, ρ) is called linearly ordered or totally ordered set if every pair of elements in S is comparable i.e., for any two elements a, b in S either $(a, b) \in \rho$ or $(b, a) \in \rho$ or $a = b$. The relation ρ is called linear order or total order relation.

Example 5.25 : The PO set $(Z, \leq)$ is a totally ordered set because for any two elements a and b either $a \leq b$ or $b \leq a$.

Example 5.26 : The PO set $(P(S), \subseteq)$ is not totally ordered set.

5.7.3 Geometrical rep

Let (S, ρ) be a finite PO set called vertex in a plane. If an a, then we place b higher th two points a and b. Thus a whose points represent the immediate predecessor/suc relation is called the Hasse properties :

1. Every vertex of S mus not necessary.
2. If a vertex b appear an edge then $a \rho b$ h is immediate prede
3. If vertex c is above then $a \rho c$ holds. H

Example 5.27 : Let order relation defi is true but 6/9 is PO set is shown b

5.8 n-arr

In Section 5.1 between two sets. elements from m

5.7.3 Geometrical representation of a PO set (Hasse Diagram)

Let (S, ρ) be a finite PO set. We represent the elements of the set S by points called vertex in a plane. If an element b is an immediate successor of the element a, then we place b higher than a in the plane and draw a line joining (edge) the two points a and b. Thus a geometrical representation or a diagram is created whose points represent the elements of the set S and the edges represent the immediate predecessor/successor relationship. This diagram of the partial order relation is called the Hasse diagram or Ordering diagram which has the following properties :

1. Every vertex of S must be related to itself, so arrows from a vertex to itself are not necessary.
2. If a vertex b appears above vertex a and if vertex a and b are connected by an edge then $a \rho b$ holds. So, the direction arrows are not necessary. Here, a is immediate predecessor of b.
3. If vertex c is above vertex a and c is connected to a by a sequence of edges then $a \rho c$ holds. Here, a is not immediate predecessor of c.

Example 5.27 : Let S = {1, 2, 3, 4, 6, 8, 9, 12, 18, 24} be a set and '/' be a partial order relation defined in S such that a/b means that a divides b. For e.g., 2/12 is true but 6/9 is not true. Here, (S, /) is a PO Set. The Hasse diagram of the PO set is shown below :

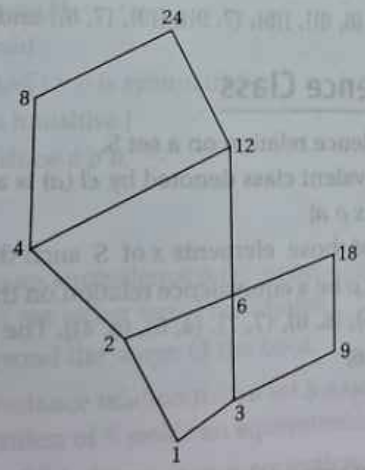


Fig. 5.1

5.8 n-array Relation

In Section 5.1 we have discussed about binary relation that is the relation between two sets. In this Section we shall discuss about the relationship among elements from more than two sets.

Let $A_1, A_1, ..., A_n$ be sets. An n-array relation on these sets is a subset of $A_1 \times A_1 \times ... \times A_n$. The sets $A_1, A_1, ..., A_n$ are called the domains of the relations and 'n' is called its degree.

Example 5.28 : Let ρ be the relation consisting of triplets (a, b, c) where a, b and c are integers with $b = a + 1$, $c = b + 1$. Then $(1, 2, 3) \in \rho$ but $(2, 4, 9) \notin \rho$. The degree of this relation is 3. Its domain are all equal to the set of integers.

Example 5.29 : Let V be an n-dimensional vector space over the field of real numbers R. Let $v_i \in V$ and $V = (v_1, v_2, ..., v_n)$. Then ρ is a defined relation among v_i's.

5.9 Partition

A partition $(S_1, S_2, ..., S_i)$ of a non-empty set S is defined as a collection of non-empty subsets of S, such that

1. Every element of S belongs to one of S_i i.e., the union of S_i is equal to set S.
2. If S_i and S_j are distinct, then $S_i \cap S_j = \phi$ *i.e.*, the partitions divide the elements of the set S into disjoint subsets.

The subsets in a partition are called blocks or cells.

Example 5.30 : Let S = {7, 8, 9}. We determine all the partitions of the set S. Here, in set S there are three elements. So, at the maximum in a partition there can be 1, 2 or 3 elements. The partitions are as follows :

[{7}, {8}, {9}], [{7}, {8, 9}], [{8}, {7, 9}], [{9}, {7, 8}] and [{7, 8, 9}]

5.10 Equivalence Class

Let P be an equivalence relation on a set S.

Let $a \in$ S. The equivalent class denoted by $cl\,(a)$ is a subset of S defined by

$cl\,(a) = \{ x \in S : x \rho a\}$

$cl\,(a)$ is the set of those elements x of S such that $x \rho a$ holds.

Example 5.31 : Let ρ be a equivalence relation on the set S = {4, 5, 6, 7} defined by ρ = {(4, 4), (5, 5), (6, 6), (7, 7), (4, 6), (6, 4)}. The equivalence classes are

$cl\,(4) = cl\,(6) = \{4, 6\}$

$cl\,(5) = cl\,(5)$

$cl\,(7) = cl\,(7)$

Let us discuss about some related theorems :

Theorem-5.1 : Let ρ be an equivalence relation on a and $a, b \in$ S, then $cl\,(a) = cl\,(b)$ iff $a \rho b$.

$cl\,(a)$ and $cl\,(b)$ are non-empty since $a \in cl\,(a)$ and $b \in cl\,(b)$

Proof : Let $cl\,(a) = cl\,(b)$ and $x \in cl\,(a)$.

Then $x \in cl\,(b)$

$\therefore x \in cl\,(a) \Rightarrow x \rho a$.

and $x \in cl(b) \Rightarrow x \rho b$ hold.
$x \rho a$ and $x \rho b$
$\Rightarrow a \rho x$ and $x \rho b$ [$\because \rho$ is symmetric] or
$\Rightarrow a \rho b$ [$\because \rho$ is transitive].
$\therefore cl(a) = cl(b) \Rightarrow a \rho b$.
Conversely, let $a, b \in S$ and $a \rho b$ holds.
Let, $x \in cl(a) \Rightarrow x \rho a$ holds.
Now, $x \rho a$ and $a \rho b$.
$\Rightarrow x \rho b$ [$\because \rho$ is transitive]
$\Rightarrow x \in cl(b)$.
$\therefore cl(a) \subseteq cl(b)$.
Similarly, $cl(b) \subseteq cl(a)$
$\therefore cl(a) = cl(b)$.
Thus, $a \rho b \Rightarrow cl(a) = cl(b)$.
Hence the theorem.

Theorem-5.2 : Let P be an equivalence relation on a set S and $a, b \in S$. If $a \rho b$, then $cl(a)$ and $cl(b)$ are disjoint.

Proof : Let $cl(a) \cap cl(b) \neq \phi$
Let, $x \in cl(a) \cap cl(b)$
$\Rightarrow x \in cl(a)$ and $x \in cl(b)$
$\Rightarrow x \rho a$ and $x \rho b$ hold
$\Rightarrow a \rho x$ and $x \rho b$ hold [$\because \rho$ is symmetric]
$\Rightarrow a \rho b$ hold [$\because \rho$ is transitive.]
This is a contradiction since $a \rho b$.
$\therefore cl(a) \cap cl(b) = \phi$
Hence the theorem.

Theorem-5.3 : Let P be an equivalence relation on a set S and $a, b \in S$. Then the classes $cl(a)$ & $cl(b)$ are either equal or disjoint.

Proof : The proof is beyond the scope of the book

Theorem-5.4 : An equivalence relation ρ on a set S determines a partition on S. Conversely, each partition of S yields an equivalence relation on S.

Proof : Each element of S belongs to $a \rho$ equivalence class if $\rho \in S$ then $\rho \in cl(\rho)$. If $a, b \in S$ then the ρ – equivalence classes $cl(a)$ and $cl(b)$ are either disjoint or equal.
We consider the family of distinct ρ-equivalence classes and we observe that

1. Each class of the family is non-empty.
2. The union of the family of the classes is the set S, and
3. The classes are pairwise disjoint.

Therefore the family of disjoint ρ-equivalence classes form a partition on the set S. Conversely, let there be a partition ρ of the set S into subset.

Let us define a relation ρ on the set S to mean that $a \rho b$ holds if a and b belong to one and the same subset of the partition ρ.

Let $a \in$ S then $a \rho a$ holds. Since a and a belong to one and the same subset of the partition ρ.

$\therefore$ ρ is reflexive.

Let $a, b \in$ S and $a \rho b$ holds. Then a and b belong to one and the same subset of the partition ρ and therefore b and a belong to one and same subset of the partition ρ.

$\therefore$ $a \rho b \Rightarrow b \rho a$.

$\therefore$ ρ is symmetric.

Let $a, b, c \in$ S and $a \rho b$, $b \rho c$ both hold. Then a and b belong to one and same subset, say S′ of ρ; b & c belong to one and same subset of ρ, say S″.

Now, since $b \in S' \cap S''$, it follows that S′ = S″ and consequently a and c belong to one and same subset of ρ. i.e., $a \rho c$ holds .

$\therefore$ $a, b, c \in$ S, $a \rho b$ and $b \rho c \Rightarrow a \rho c$.

$\therefore$ P is transitive.

Thus ρ is an equivalence relation on S.

Hence the theorem.

5.11 Congruence Modulo Relation

Let a and b be two integers and m be a positive integer. Then a is said to be congruent to b modulo m, if $a - b$ is divisible my m, *i.e.*, $\frac{a-b}{m}$ is an integer.

We write $a \equiv b \pmod{m}$. Here, m is called the modulus of the congruence, and b is called a residue of $a \pmod{m}$

Then, $a - b = km$, *i.e.*, $a = b + km$ for some integer k.

Note

If $a - b$ is not divisible by m, we say that a is not congruent to b modulo m and we write it as $a \not\equiv b \pmod{m}$

Example 5.32 : We have $50 - 10 = 40$ is divisible by 5, so $50 \equiv 10 \pmod{5}$. Here 5 is the modulus of the congruent and 10 is the residue of 50((mod 5)

Example 5.32 : Since $32 - 3 = 29$ is not divisible by 7 so $32 \not\equiv 3 \pmod{7}$.

Theorem-5.5 : 'Congruence Modulo m' is an equivalence relation on the set of all integers

Proof : Consider Z = set of all integers and the relation 'congruence modulo m' Written as $a \equiv b \pmod{m}$ between a and b, for the positive integer m.

Now, $\frac{a-a}{m} = 0$ which is an inte

So, the relation 'congruence mo

Now, $a \equiv b \pmod{m} \Rightarrow \frac{a-b}{m}$ =

$\Rightarrow \frac{b-a}{m}$ = some intege

$\Rightarrow b \equiv a \pmod{m}$

So, the relation 'congruence r

Now, let $a \equiv b \pmod{m}$ and b

Thus, $a - b = mk$ and $b - c =$

Adding the two terms, we ge

$a - c = m(k + r)$

$\Rightarrow \frac{a-c}{m} = k + r$ which is

$\Rightarrow a \equiv c \pmod{m}$

So, the relation 'congruence

Hence, the relation 'congru

Since, the relation 'congru

set of integers Z, so this rel

theorem we find this partit

Theorem-5.6 : The partiti

$\{[0], [1], [2] \ldots [m-1]\}$

Proof : If $a \in$ Z then the e

$[a] = \{x : x \in$ Z and $a \equiv x$ (n

By property of equivalence

$[a] = [b]$ or, $[a] \cap [b] = \phi$

Now, we shall show $[0] \cup$

Obviously the left hand si

an element of Z. Converse

Now, if $0 \le a \le m - 1$ the

Obviously then $a \in [0] \cup$

But if $a > m - 1$ or $a < 0$ v

$a = mp + r$ (r is the remair

So, $\frac{a-r}{m}$ = P showing that

$\Rightarrow r \equiv a \pmod{m}$, or, a

Since $[r]$ is one of $[0], [1],$

So, $a \in [0] \cup [1] \cup [2] \cup \ldots$

Now, $\frac{a-a}{m} = 0$ which is an integer

So, the relation 'congruence modulo m' is reflexive

Now, $a \equiv b \pmod{m} \Rightarrow \frac{a-b}{m}$ = some integer

$\Rightarrow \frac{b-a}{m}$ = some integer again

$\Rightarrow b \equiv a \pmod{m}$

So, the relation 'congruence modulo m' is symmetric

Now, let $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$

Thus, $a - b = mk$ and $b - c = mr$ where k and r are integers.

Adding the two terms, we get

$a - c = m(k + r)$

$\Rightarrow \frac{a-c}{m} = k + r$ which is an integer

$\Rightarrow a \equiv c \pmod{m}$

So, the relation 'congruence modulo m' is transitive

Hence, the relation 'congruence modulo m' is an equivalence relation.

Since, the relation 'congruence modulo m' is an equivalence relation on the set of integers Z, so this relation will give a partition on Z. From the following theorem we find this partition of Z.

Theorem-5.6 : The partition given by the relation 'congruence modulo m' is $\{[0], [1], [2] \ldots [m-1]\}$

Proof : If $a \in Z$ then the equivalence class

$[a] = \{x : x \in Z \text{ and } a \equiv x \pmod{m}\}$

By property of equivalence class we see for any two integers a and b, that either $[a] = [b]$ or, $[a] \cap [b] = \phi$

Now, we shall show $[0] \cup [1] \cup [2] \ldots [m-1] = Z$

Obviously the left hand side is a proper subset of Z as every element of $[a]$ is an element of Z. Conversely let $a \in Z$

Now, if $0 \le a \le m-1$ the class $[a]$ is any of $[0]$, $[1]$, $[2]$, ... and $[m-1]$.

Obviously then $a \in [0] \cup [1] \cup \ldots \cup [m-1]$.

But if $a > m-1$ or $a < 0$ we can find integers p and r, $0 \le r \le m-1$ such that $a = mp + r$ (r is the remainder when a is divided by m)

So, $\frac{a-r}{m} = P$ showing that $a \equiv r \pmod{m}$

$\Rightarrow r \equiv a \pmod{m}$, or, $a \in [r]$

Since $[r]$ is one of $[0]$, $[1]$, ..., $[m-1]$

So, $a \in [0] \cup [1] \cup [2] \cup \ldots \cup [m-1]$.

Thus, $Z \subset [0] \cup [1] \cup [2] \cup ... \cup [m-1]$.

Hence, $[0] \cup [1] \cup [2] \cup ... \cup [m-1] = Z$

So, $\{[0] \cup [1] \cup [2] \cup ... \cup [m-1]\}$ is a partition of Z.

Thus each of the equivalence classes [0], [1], ..., $[m-1]$ is called *a* 'Residue class modulo m'.

Example 5.34 : There are four residue class modulo 4 which are

$[0] = \{..., -12, -8, -4, 0, 8, 12, 16, ...\}$

$[1] = \{..., -11, -7, -3, 1, 5, 9, 13, 17, ...\}$

$[2] = \{..., -10, -6, -2, 2, 6, 10, 14, 18, ...\}$

$[3] = \{..., -9, -5, -1, 3, 7, 11, 15, 19, ...\}$

Any two of the above four classes are disjoint and the union of the four classes is Z. It is to be noted that $[4] = [0]$, $[5] = [1]$, $[6] = [2]$, ... and so on.

5.12 Composition of Relation

Let ρ and σ be relation on a set S. Then the composite relation $\sigma \circ \rho$ is defined by $\sigma \circ \rho = \{(a, c) : a \in S, c \in S\}$ if there is element b in S such that $(a, b) \in \rho$ and $(b, c) \in \sigma$.

Example 5.35 : Let $A = \{2, 3, 5, 6, 7\}$ and $\rho = \{(2, 3), (2, 5), (3, 3), (5, 5), (6, 6)\}$, $\sigma = \{(3, 2), (5, 3), (6, 5), (6, 6)\}$.

Then, $\sigma \circ \rho = \{(2, 2), (2, 3), (3, 2), (5, 3), (6, 5), (6, 6)\}$

$\rho \circ \sigma = \{(3, 3), (3, 5), (5, 3) (6, 5), (6, 6)\}$.

Notes

Let ρ be a relation on a set S. Then ρ is an equivalence relation on ρ if and only if.

1. $\delta_S \subset \rho$, where $\delta_S = \{(a, a) : a \in S\}$; 2. $\rho^{-1} = \rho$ and 3. $\rho \circ \rho \subset \rho$.

Theorem-5.7 :

1. The inverse of an equivalence relation is an equivalence relation.
2. The intersection of two equivalence relations on a set is an equivalence relation.
3. The union of two reflexive relations on a set is a reflexive relation.
4. The union of two symmetric relations on set is a symmetric relation.

◆ **Proof :** Left as exercise

Illustra

1. **Let R be a relation on**
$R = \{(k, l), (m, l), (n,$
Find domain and ra

Solution The domain of
The range of relation R i

2. **Determine wheth**
plane defined by
equivalence relat

Solution **Reflexive Pro**
for every $a \in A$. Therefore

Symmetric Property :
is also similar to *a*. Hence
R is symmetric.

Transitive Property :
We have triangle *a* si
Hence, whenever $(a,$
Therefore, R is transi
So, being reflexive,
relation on the set A.

3. **Show that an**
of S. Is the co

Solution Each elem
then $p \in cl(p)$. If a, b
$cl(b)$ are either disjoi
If we consider the
(i) each class in
(ii) the union of
(iii) the classes a
Therefore the fam
set S.

Conversely, let th
Let us define a rel
to one and same sub
Let $a \in S$. The a
the partition P. Thus

Illustrative Examples

1. **Let R be a relation on a set A = {k, l, m, n} defined by R = {(k, l), (m, l), (n, l), (l, l), (k, k), (m, k), (l, k), (n, k)} Find domain and range of relation .**

Solution The domain of relation R is Dom (R) = {k, l, m, n} = A
The range of relation R is ran (R) = {k, l}

2. **Determine whether the relation R on the set A of all triangles in the plane defined by R = {(a, b) : triangle a is similar to triangle b} is an equivalence relation.**

Solution **Reflexive Property** : Triangle a is similar to b itself. Hence $(a, a) \in R$ for every $a \in A$. Therefore, R is reflexive.

Symmetric Property : If triangle a is similar to b , then it is sure that triangle b is also similar to a. Hence, whenever $(a, b) \in R$, we have (b, a) also belong to R. So, R is symmetric.

Transitive Property : If triangle a is similar to b and b is similar to c.
We have triangle a similar to c.
Hence, whenever $(a, b) \in R$ and $(b, c) \in R$, we have $(a, c) \in R$.
Therefore, R is transitive.
So, being reflexive, symmetric and transitive; the relation R is an equivalence relation on the set A.

3. **Show that an equivalence relation ρ on a set S determines a partition of S. Is the converse true? Justify.** [KU, 2019]

Solution Each element of S belongs to a ρ-equivalence class, because if $p \in S$ then $p \in cl(p)$. If a, b be elements of S then the ρ–equivalence classes $cl(a)$ and $cl(b)$ are either disjoin or equal.
If we consider the family of distinct ρ-equivalence classes, we observe that
(i) each class in the family is non-empty.
(ii) the union of the family of classes is the set S, and
(iii) the classes are pairwise disjoin.
Therefore the family of distinct ρ–equivalence classes form a partition of the set S.

Conversely, let there be a partition P of the set S into subsets.
Let us define a relation ρ on the set S to mean that $a \rho b$ holds if a and b belongs to one and same subset of the partition P.
Let $a \in S$. The $a \rho a$ holds since a and b belong to one and the same subset of the partition P. Thus ρ is reflexive.

Let $a, b \in R$ and $a \rho b$. Then a and b belong to one and the same subset of the partition P and therefore b and a belong to the same subset of the partition P, i.e. $a \rho b \Rightarrow b \rho a$. Therefore ρ is symmetric.

Let $a, b, c \in S$ and $a \rho b$, $b \rho c$ both hold. Then a and b belong to one and the same subset, say S′ of P; b and c belong to one and the same subset, say S″ of P.

S′ and S″ being subsets of a partition, must be either identical or disjoint. Since $b \in S' \cap S''$, it follows that $S' = S''$ and consequently, a and c belong to one and the same subset of P. That is, $a \rho b$ and $b \rho c \Rightarrow a \rho c$. Therefore ρ is transitive.

Thus ρ is an equivalent relation on S.

4. Determine whether the relation $S = \{(a, b) : a \geq b\}$ on the set R of real Numbers is an equivalence relation.

Solution **Reflexive Property :** We have $(a, a) \in S$, for every $a \in R$ because $a = a$ for every $a \in R$

Hence, the relation is reflexive.

Symmetric Property : Let $(a, b) \in S$. So, we have $a \geq b$ but for this it is not possible that $b \geq a$.

Hence, $(b, a) \notin S$. Therefore, relation is not symmetric.

Transitive Property : Let $(a, b) \in S$ and $(b, c) \in S$. So, when we have $a \geq b$ and $b \geq c$, it is sure that $a \geq c$.

Hence, $(a, c) \in S$. Therefore, the relation is transitive.

So, being the relation reflexive and transitive but not symmetric, it is not an equivalence relation.

5. Let $\mathbb{Z}$ be the set of all integers. If $a, b \in \mathbb{Z}$, then $a \equiv b \pmod 5$ if and only if $a - b$ has factor 5. Then show that '$\equiv$' is an equivalent relation. [KU, 2019]

Solution Let $a \in \mathbb{Z}$, Then $a - a$ has a factor 5. Thus '$\equiv$' is reflexive.

Let $a, b \in \mathbb{Z}$ and $a \equiv b \pmod 5$ hold. Then $a - b$ has a factor 5. Therefore $b - a$ has also a factor 5. Thus '$\equiv$' is symmetric.

Let $a, b, c \in \mathbb{Z}$ and $a \equiv b \pmod 5$, $b \equiv c \pmod 5$ both hold. Then $(a - b)$ and $(b - c)$ both have 5 as factor. So $a - c = (a - b) + (b - c)$ has also factor 5. Thus '$\equiv$' is transitive.

So combining all above, we can conclude that '$\equiv$' is an equivalent relation on $\mathbb{Z}$.

6. Prove that the relation R "$a - b$ is divisible by 5" $\forall a, b \in I^+$ (a set of positive integers) is an equivalence relation.

Solution **Reflexive Property :** The relation is reflexive because $a - a$ is divisible by 5 for every $a \in I^+$.

Symmetric Property : Let $(a, b) \in R$. If $a - b$ is divisible by 5 then it is sure that $b - a$ is also divisible by 5.

Hence $(b, a) \in R$. Therefore, the relation is symmetric.

Transitive Property : Let $(a, b) \in R$ and $(b, c) \in R$
i.e., $a - b$ is divisible by 5.
$b - c$ is divisible by 5.
It implies $a - c$ is divisible by 5.
Hence, $(a, c) \in R$
Therefore, the relation is transitive.
So, being reflexive, symmetric and transitive, the relation "$a - b$ is divisible by 5" is equivalence relation.

7. Give an example of a relation which is reflexive and symmetric but not transitive. [KU, 2019]

Solution Let us define a set $A = \{1, 2, 3\}$ and we define a relation R on A, where $R = \{(1, 1), (1, 2), (2, 1), (2, 2), (2, 3), (3, 2), (3, 3)\}$.

Since $(1, 1), (2, 2), (3, 3) \in A$ and $1, 2, 3 \in A$. Thus, relation R is reflexive.

Again, $(1, 2) \in R$ and also $(2, 1) \in R$, where $1, 2 \in R$. Similarly, $(2, 3) \in R$ and also $(3, 2) \in R$, where $2, 3 \in A$. Thus relation R is also symmetric.

Here $(1, 2) \in R$ and $(2, 3) \in R$, $1, 2, 3 \in A$. But $(1, 3) \notin R$. Therefore, R is not transitive.

So, this is an example of relation which is reflexive and symmetric but not transitive.

8. Let us assume that ρ is a relation on the set R of real numbers defined by $x \rho y$ if and only if $x - y$ is an integer. Prove that ρ is an equivalence relation on R.

Solution **Reflexive Property :** Consider x belongs to R, then $x - x = 0$ which is an integer. Therefore, $x \rho x$.

Symmetric Property : Consider x and y belongs to R and $x \rho y$. Then $x - y$ is an integer. Thus, $y - x = -(x - y)$, $y - x$ is also an integer. Therefore, $y \rho x$.

Transitive Property : Consider x and y belongs to R, $x \rho y$ and $y \rho z$. Therefore $x - y$ and $y - z$ are integers. According to the transitive property,
$(x - y) + (y - z) = x - z$ is also an integer. So, $x \rho z$.
Thus, ρ is an equivalence relation on R.

9. A relation ρ is defined on the set Z by $a \rho b$ if and only if $2a + 3b$ is divisible by 5 $\forall a, b \in Z$. Show that ρ is an equivalence relation. [VU 2018]

Solution Let $a \in \mathbb{Z}$ so, $2a + 3a = 5a$ which is divisible by 5.
Since a is arbitrary so $a \rho a$ holds for all $a \in \mathbb{Z}$
So ρ is a reflexive relation.
Now let $a, b \in \mathbb{Z}$ and $a \rho b$ holds that is $2a + 3b$ is divisible by 5
So, there exist an integer m such that $2a + 3b = 5m$

Let $n = 2b + 3a$

$n + 5m = 2b + 3a + 2a + 3b = 5(a + b)$

$n = 5(a + b - m)$ here $a + b - m \in Z$

So, n is divisible by 5 that is $2b + 3a$ is divisible by 5

Thus, $b \rho a$ holds

So, ρ is a symmetric relation.

Now let $a, b, c \in \mathbb{Z}$ and $a \rho b$ and $b \rho c$ holds

i.e., $2a + 3b$ and $2b + 3c$ are divisible by 5

So, there exist integers p and q such that $2a + 3b = 5p$ and $2b + 3c = 5q$

Adding these two, we get $2a + 3b + 2b + 3c = 5p + 5q$

$$2a + 3c = 5(p + q - b)$$

$a \rho c$ holds

ρ is a transitive relation

So, ρ is a equivalence relation.

10. **Let A be a set with 2 elements. How many reflexive relation can be defined on A.** [VU 2018]

Solution We know that if a set A have n elements, then there are 2^{n^2-n} reflexive relation can be defined on A.

So here we can define $2^{2^2-2} = 2^2 = 4$ reflexive relations on A.

11. **Show that the relation ρ is an equivalence relation in the set S = {1, 2, 3, 4, 5} given by the relation $\rho = \{(a, b) : |a - b| \text{ is even}\}$.**

Solution $R = \{(a, b) : |a - b| \text{ is even}\}$. Where a, b belongs to S

Reflexive Property: From the given relation, $|a - a| = |0| = 0$

Thus, $|a - a|$ is even, therefore, (a, a) belongs to R

Hence ρ is Reflexive

Symmetric Property : From the given relation, $|a - b| = |b - a|$

We know that $|a - b| = |-(b - a)| = |b - a|$

Hence $|a - b|$ is even, then $|b - a|$ is also even.

Therefore, if $(a, b) \in \rho$, then (b, a) belongs to ρ

Hence ρ is symmetric.

Transitive Property : If $|a - b|$ is even, then $(a - b)$ is even.

Similarly, if $|b - c|$ is even, then $(b - c)$ is also even.

Sum of even numbers are also even

So, we can write it as $a - b + b - c$ is even

Then, $a - c$ is also even

So, $|a - b|$ and $|b - c|$ are even, then $|a - c|$ is even.

Therefore, if $(a, b) \in \rho$ and $(b, c) \in \rho$, then (a, c) also belongs to ρ

Hence ρ is transitive.

12. **Let ρ and ρ' be two ... an equivalence rela...**

Solution Let ρ and ρ' be ...

Since both ρ and ρ' are ...

So, $(a, a) \in \rho \cap \rho'$ for all ...

$\rho \cap \rho'$ is reflexive.

Now let $a, b \in A$ and $(a, ...$

i.e., $(a, b) \in \rho$ and $(a, b) ...$

i.e., $(b, a) \in \rho$ and $(b, a) ...$

So, $(b, a) \in \rho \cap \rho'$

$\rho \cap \rho'$ is symmetric.

Now let $a, b, c \in A$ and ...

i.e., $(a, b) \in \rho$ and $(b, c) ...$

$\Rightarrow (a, c) \in \rho$.

Similarly, $(a, c) \in \rho'$

So, $(a, c) \in \rho \cap \rho'$

$\rho \cap \rho'$ is transitive.

Hence $\rho \cap \rho'$ is an eq...

13. **Let R and S b... (ii) $(R \cup S)^{-1} = ...$**

Solution **(i)** Let $(a, b) ...$

Now $(b, a) \in R$ and ...

This means $(a, b) \in ...$

Hence, $(a, b) \in R^{-1} \cap ...$

Conversely, let $(a, b) ...$

This means $(b, a) \in ...$

Hence, $(a, b) \in (R \cap ...$

From (1) and (2), we ...

Hence proved.

(ii) Let $(a, b) \in (R \cup S ...$

Now, $(b, a) \in R$ or ...

Hence, $(a, b) \in R^{-} ...$

Conversely, let $(a, ...$

This means $(b, a) ...$

Hence, $(a, b) \in (R ...$

From (1) and (2), ...

Hence proved.

Let ρ and ρ' be two equivalence relations, then show that $\rho \cap \rho'$ is also an equivalence relation. [VU 2018]

Solution Let ρ and ρ' be two equivalence relation on A.

Since both ρ and ρ' are reflexive, we have $\forall a \in A$. $(a, a) \in \rho$ and $(a, a) \in \rho'$

So, $(a, a) \in \rho \cap \rho'$ for all $a \in A$.

$\rho \cap \rho'$ is reflexive.

Now let $a, b \in A$ and $(a, b) \in \rho \cap \rho'$

i.e., $(a, b) \in \rho$ and $(a, b) \in \rho'$

i.e., $(b, a) \in \rho$ and $(b, a) \in \rho'$

So, $(b, a) \in \rho \cap \rho'$

$\rho \cap \rho'$ is symmetric.

Now let $a, b, c \in A$ and $(a, b) \in \rho \cap \rho'$ and $(b,c) \in \rho \cap \rho'$

i.e., $(a, b) \in \rho$ and $(b, c) \in \rho$.

$\Rightarrow (a, c) \in \rho$.

Similarly, $(a, c) \in \rho'$

So, $(a, c) \in \rho \cap \rho'$

$\rho \cap \rho'$ is transitive.

Hence $\rho \cap \rho'$ is an equivalence relation.

Let R and S be two relations. Show that (i) $(R \cap S)^{-1} = R^{-1} \cap S^{-1}$ (ii) $(R \cup S)^{-1} = R^{-1} \cup S^{-1}$.

Solution (i) Let $(a, b) \in (R \cap S)^{-1}$. So, we have $(b, a) \in (R \cap S)$.

Now $(b, a) \in R$ and $(b, a) \in S$

This means $(a, b) \in R^{-1}$ and $(a, b) \in S^{-1}$.

Hence, $(a, b) \in R^{-1} \cap S^{-1}$ ... (1)

Conversely, let $(a, b) \in R^{-1} \cap S^{-1}$. So, we have $(a, b) \in R^{-1}$ and $(a, b) \in S^{-1}$.

This means $(b, a) \in R$ and $(b, a) \in S$. So, $(b, a) \in R \cap S)$

Hence, $(a, b) \in (R \cap S)^{-1}$ ... (2)

From (1) and (2), we have $(R \cap S)^{-1} = R^{-1} \cap S^{-1}$

Hence proved.

(ii) Let $(a, b) \in (R \cup S)^{-1}$. So, we have $(b, a) \in (R \cup S)$

Now, $(b, a) \in R$ or $(b, a) \in S$. This means $(a, b) \in R^{-1}$ or $(a, b) \in S^{-1}$

Hence, $(a, b) \in R^{-1} \cup S^{-1}$ ... (1)

Conversely, let $(a, b) \in R^{-1} \cup S^{-1}$. So, we have $(a, b) \in R^{-1}$ or $(a, b) \in S^{-1}$.

This means $(b, a) \in R$ or $(b, a) \in S$. So, $(b, a) \in (R \cup S)$.

Hence, $(a, b) \in (R \cup S)^{-1}$ ... (2)

From (1) and (2), we have $(R \cup S)^{-1} = R^{-1} \cup S^{-1}$.

Hence proved.

14. **Give an example of a relation which is symmetric but not reflexive and transitive.** [VU 2018]

Solution Let A = {1, 2, 3, 4}

ρ is defined as $\rho = \{(1, 1), (1, 2), (2,1), (2, 3), (3, 2), (2, 2)\}$

Here ρ is not reflexive because $3 \rho 3$ does not hold.

ρ is not transitive because $1 \rho 2$ and $2 \rho 3$ hold but $1 \rho 3$ does not hold.

But ρ is symmetric because $1 \rho 2$ and $2 \rho 1$ hold

Similarly $2 \rho 3$ and $3 \rho 2$ holds.

15. **Give an example of a relation which is reflexive and symmetric but not transitive.** [BU 2019]

Solution Let us define a set A = {1, 2, 3} and we define a relation R on A, where R = {(1, 1), (1, 2), (2, 1), (2, 2), (2, 3), (3, 2), (3, 3)}

Since (1, 1), (2, 2), (3, 3) $\in$ A, relation R is reflexive.

Again, (1, 2) $\in$ R and also (2, 1) $\in$ R, where 1, 2 $\in$ A. Similarly, (2, 3) $\in$ R and also (3, 2) $\in$ R, where 2, 3 $\in$ A. Thus relation R is also symmetric.

Here (1, 2) $\in$ R and (2, 3) $\in$ R, 1, 2, 3 $\in$ A. But (1, 3) $\notin$ R. Therefore, R is not transitive.

So, this is an example of relation which is reflexive and symmetric but not transitive.

16. **Give an example of relations R_1, R_2, R_3, R_4 and R_5 on A = {4, 5, 6, 7, 8} having property**

(i) R_1 is reflexive and transitive but not symmetric.

(ii) R_2 is symmetric and antisymmetric.

(iii) R_3 is antisymmetric but not reflexive.

(iv) R_4 is neither symmetric nor antisymmetric.

(v) R_5 is neither symmetric and asymmetric nor antisymmetric.

Solution (i) $R_1 = \{(4, 4), (5, 5), (6, 6), (7, 7), (8, 8), (4, 5), (5, 6), (4, 6)\}$

(ii) $R_2 = \{(4, 4), (5, 5), (6, 6), (7, 7)\}$

(iii) $R_3 = \{(4, 4), (5, 5), (6, 6)\}$

(iv) $R_4 = \{(4, 4), (4, 5), (4, 6)\}$

(v) $R_5 = \{(4, 4), (4, 5), (5, 4), (4, 6)\}$

17. **Determine the nature of the following relation R on the set $\mathbb{Z}$.**

A. a R b if and only if $a, b \in \mathbb{Z}$ and $ab \geq 0$

B. a R b if and only if $a, b \in \mathbb{Z}$ and $|a - b| \leq 3$ [VU, 2018]

Solution **A.** Let $a \in \mathbb{Z}$, $a \cdot a = a^2 \geq 0$ for all $a \in \mathbb{Z}$

a R a holds for all $a \in \mathbb{Z}$

R is a reflexive relation.
Let $a, b \in \mathbb{Z}$ and $a\,R\,b$ holds that is $a \cdot b \geq 0$ imples that $ba \geq 0$
$b\,R\,a$ holds.
R is symmetric relation.
Let $a, b, c \in \mathbb{Z}$ and $a\,R\,b$ and $b\,R\,c$ holds that is $ab \geq 0$ and $bc \geq 0$
$ab^2c \geq 0$ since $b^2 \geq 0$ for all $b \in \mathbb{Z}$.
$ac \geq 0$
$a\,R\,c$ holds
R is a transitive relation.
Thus, R is an equivalance relation.

8. Let $a \in \mathbb{Z}$. $|a - a| = 0 \leq 3$ for all $a \in \mathbb{Z}$
$a\,R\,a$ holds for all $a \in \mathbb{Z}$.
R is a reflexive relation.
Let $a, b \in \mathbb{Z}$ and $a\,R\,b$ holds, that is $|a - b| \leq 3$
imples that $|b - a| \leq 3$
$b\,R\,a$ holds.
R is symmetric relation.
For $1, 3, 5 \in \mathbb{Z}$. $1\,R\,3$ holds since $|1 - 3| = 2 \leq 3$ and $3\,R\,5$ holds since $|3 - 5| = 2 \leq 3$
But $|1 - 5| = 4 > 3$. that is $1\,R\,5$ does not hold.
R is not a transitive relation.
So, R is not a equivalance relation.

9. Give an example of relations R_1, R_2, R_3 and R_4 on $A = \{a, b, c, d\}$ having Property
(i) R_1 is irreflexive and antisymmetric
(ii) R_2 is asymmetric and antisymmetric.
(iii) R_3 is asymmetric but $R_2 \cup R_3^{-1}$ is symmetric.
(iv) R_4 is transitive but $R_4 \cup R_4^{-1}$ is not transitive.

Solution (i) $R_1 = \{(a, b), (b, c), (c, d), (d, a)\}$
(ii) $R_2 = \{(a, a), (b, b), (c, d), (b, d), (a, d)\}$
(iii) $R_3 = \{(a, b), (b, c), (c, d), (d, a)\}$
(iv) $R_4 = \{(a, c), (d, d)\}$

10. Let S be a set containing three elements. How many different binary relations can be defined on S. [VU, 2019]

Solution For a set S containing three elements, number of binary relations that can be defined on S are $2^{3^2} = 2^9 = 512$

20. **Give an example of a relation R on the set A = {4, 5, 6, 7} which have no property i.e., the relation R is not reflexive, irreflexive, symmetric, asymmetric, transitive and antisymmetric.**

Solution The relation having no property is R = {(4, 4), (5, 5), (4, 5), (5, 4), (4, 7)}.

21. **Give an example of antisymmetric relation.** [VU, 2019]

Solution Let A = {1, 2, 3, 4} and let R be a relation on A defined by

R = {(1, 1), (1, 2), (2, 3), (3, 4),(4, 4)}

Here R is an antisymmetric relation.

22. **Define *n*-ary relation.** [VU 2020]

Solution An *n*-ary relation on set $A_1, A_2 \ldots A_n$ is a set of ordered *n*-tuples $(a_1, a_2, \ldots a_n)$ where a_i is an element of A_i for all $i.1 \le i \le n$.

Thus an *n*-ary relation on sets $A_1, A_2 \ldots A_n$ is a subset of Cartesian product $A_1 \times A_2 \times \ldots \times A_n$.

23. **Let S = {1, 2, 3, 4, 5, 6, 7, 8}. Determine whether the following is a partition of S or not.**

(i) $P_1 = [\{1, 2, 3, 4\}, \{1, 3, 5, 6, 7, 8\}]$

(ii) $P_2 = [\{1, 3, 5, 7\}, \{2, 4\}, \{6, 8\}]$

(iii) $P_3 = [\{1, 2, 3, 4\}, \{6, 8\}]$

(iv) $P_1 = [\{1, 3, 5, 6, 7, 8\}, \{2, 4\}]$

Solution **(i)** P_1 is not a partition as $1 \in S$ belongs to both the cells

(ii) P_2 is a partition

(iii) P_3 is not a partition as $7 \in S$ does not belong to any cells of P_3

(iv) P_4 is a partition

24. **Determine the nature of the following relation R on the set $\mathbb{Z}$. Where a R b if and only if $a - b$ is divisible by 5.** [VU 2020]

Solution Let $a \in \mathbb{Z}$, so $a - a = 0$ which is divisible by 5.

Since a is arbitrary so a R a holds for all $a \in \mathbb{Z}$

So, R is a reflexive relation.

Now, Let $a, b \in \mathbb{Z}$ and a R b holds, that is, $a - b$ is divisible by 5

Then, there exists an integer m such that $a - b = 5m$

$b - a = 5(-m)$ here $-m \in \mathbb{Z}$

So, $b - a$ is divisible by 5

b R a holds

So, R is a symmetric relation.

Now let $a, b, c \in \mathbb{Z}$ ar
That is $a - b$ and $b -$
that $a - b = 5p$ and $b -$
Adding this two we
$a - c = 5(p + q)$ w
a R c holds
∴ R is a transitive r
So, R is a equivalen

25. **Show that the**
Is it totally or
successor or

Solution Let Q be
antisymmetric and tr
Also for any two ra
So, $(Q, \le)$ is totally

26. **Define Unive**

Solution For a set
For a set A, ϕ is ca

27. **Let (N, ρ) is**
stands for d
totally (line
(i) {2, 4, 8, 3

Solution The set
either $a \rho b$ or $b \rho a$.
The set {3, 15, 20}

28. **Show that t**
partial orde

Solution Here P(S
Let A ∈ P(S) A ⊆
So, '⊆' is reflexive
Let A, B ∈ P(S) ar
i.e., A = B from th
'⊆' is antisymetri
Finally let, A, B,
'⊆' is transitive.
'⊆' is a partially

Now let $a, b, c \in \mathbb{Z}$ and $a\,R\,b$ and $b\,R\,c$ holds
That is $a-b$ and $b-c$ is divisible by 5 that is there exist integers p and q such that $a-b=5p$ and $b-c=5q$
Adding this two we get $a-b+b-c=5p+5q$
$a-c=5(p+q)$ where $p+q \in \mathbb{Z}$
$a\,R\,c$ holds
$\therefore$ R is a transitive relation
So, R is a equivalence relation on the set Z.

25. Show that the set of all rational numbers with usual order '≤' is a PO set. Is it totally ordered? Prove that no element in this set has an immediate successor or an immediate predecessor.

Solution Let Q be the set of rational numbers. Obviously '≤' is reflexive, antisymmetric and transitive. So, Q is a PO set with '≤' as a partial order relation.
Also for any two rational numbers a and b either $a \le b$ or $b \le a$.
So, $(Q, \le)$ is totally ordered.

26. Define Universal relation and empty relation. [VU 2020]

Solution For a set A, $A \times A$ is called universal relation on A.
For a set A, ϕ is callled empty relation on A.

27. Let (N, ρ) is a PO set where N stands for set of natural numbers and ρ stands for divisibility. State which of the following two subsets of N are totally (linearly) ordered ?
(i) {2, 4, 8, 32} (ii) {3, 15, 20}

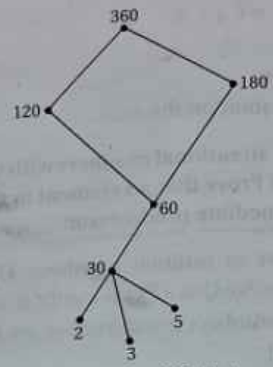
Solution The set {2, 4, 8, 32} is totally ordered as for any two elements a and b either $a\,\rho\,b$ or $b\,\rho\,a$.
The set {3, 15, 20} is not totally ordered as 3 does not divide 20.

28. Show that the relation '⊆' (Subset) defined on the power set P(S) is a partial ordered relation. [VU 2018]

Solution Here P(S) is the power set of S.
Let $A \in P(S)$ $A \subseteq A$ for all $A \in P(S)$.
So, '⊆' is reflexive.
Let $A, B \in P(S)$ and $A \subseteq B$ and $B \subseteq A$.
i.e., $A = B$ from the definition of equality of set.
'⊆' is antisymetric.
Finally let, $A, B, C \in P(S)$ and $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$
'⊆' is transitive.
'⊆' is a partially ordered set.

29. **The set X = {2, 3, 5, 30, 60, 120, 180, 360} forms a P. O Set with respect to the 'divide' relation. Draw the Hasse diagram.**

Solution The Hasse diagram of set X = {2, 3, 5, 30, 60, 120, 180, 360} with respect to the 'divide' relation is

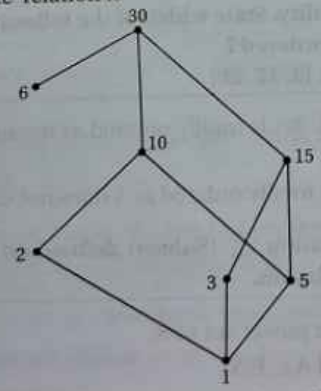


30. **Draw the Hasse diagram for the P.O. Set**

(i) A = set of all factors of 30 (including 1 and 30) with respect to 'divide' relation

(ii) A = set of all factors of 1 with respect to 'divide' relation

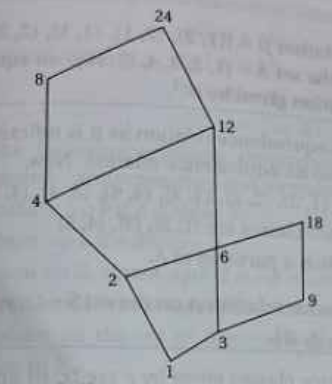
Solution The Hasse diagram of set A = set of all factors of 30 (including 1 and 30) with respect to the 'divide' relation is



(ii) The Hasse diagram of A = set of all factors of 1 with respect to 'divide' relation

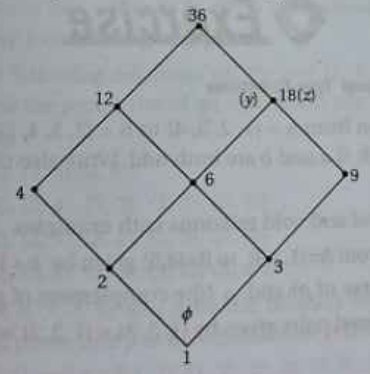
31. **Let X = {24, 18, 12, 9, 8, 6, 4, 3, 2, 1} be ordered by the relation 'x divides y'. Find the Hasse diagram.**

Solution The Hasse diagram of the set X = {24, 18, 12, 9, 8, 6, 4, 3, 2, 1} by the relation 'x divides y' is



32. **Let D_{36} be the set of all positive divisors of 36. Draw the Hasse diagram of the P.O. Set D_{36}.**

Solution The Hasse diagram of the P.O. Set D_{36} is



33. **Obtain the equivalence classes determined by the equivalence relation $\rho = \{(1, 2), (2, 1), (1, 1), (2, 2), (3, 3), (4, 4)\}$ on the set A = {1, 2, 3, 4}. Hence find the partition of A.**

Solution First take the element 1, we find that $1 \rho 2$, $2 \rho 1$ and $1 \rho 1$ hold.
Only the element 2 is related to 1.
So, one equivalence class is {1, 2} = [1]

Again 3 is related to 3. So, another equivalence class is $\{3\} = [3]$ say.
Similarly, 4 is related to 4. So, we get another equivalence class $\{4\} = [4]$
We find that $[1] \cup [3] \cup [4] = A$.
So, $[1]$, $[3]$ and $[4]$ are all the equivalence classes and the partition of A is $\{[1], [3], [4]\}$

34. **Consider the relation $\rho = \{(1, 2), (2, 1), (1, 1), (2, 2), (3, 3), (4, 4), (4, 5), (5, 4), (5, 5)\}$ on the set $A = \{1, 2, 3, 4, 5\}$. Is ρ an equivalence relation? If so find the partition given by ρ^{-1}.**

Solution Here, ρ is an equivalence relation as it is reflexive, symmetric as well as transitive. So, ρ^{-1} is also an equivalence relation. Now,

$\rho^{-1} = \{(2, 1), (1, 1), (1, 2), (2, 2), (3, 3), (4, 4), (5, 4), (4, 5), (5, 5)\}$

The distinct equivalence classes are $\{1, 2\}$, $\{3\}$, $\{4, 5\}$

Hence, $\{\{1, 2\}, \{3\}, \{4, 5\}\}$ is a partition of A.

35. **Find the equivalence relation ρ on the set $S = \{x, y, z, w\}$ which gives the partition $\{\{x, w\}, \{y, z\}\}$.**

Solution The equivalence classes given by ρ are $\{x, w\}$ and $\{y, z\}$

So, $x \rho w$ and $w \rho x$, $y \rho z$ and $z \rho y$

So, $\rho = \{(x, w), (w, x), (y, z), (z, y)\}$

Exercise

A Short and Long Answer Type Questions

1. If ρ is the relation from $A = \{1, 2, 3, 4\}$ to $B = \{2, 3, 4, 5\}$, list the elements in ρ, defined by $a \rho b$, if a and b are both odd. Write also the domain and range of ρ.
2. 2. Define universal and void relations with examples.
3. If ρ is a relation from $A=\{1,2,3\}$ to $B=\{4,5\}$ given by $\rho = \{(1,4),(2,4),(1,5),(3,5)\}$, find ρ^{-1} (the inverse of ρ) and $\bar{\rho}$ (the complement of ρ).
4. Which of the ordered pairs given by $\{1, 2, 3\} \times \{1, 2, 3\}$ belong to the following relations :
   (i) $a \rho b$ iff $a \leq b$, (ii) $a \rho b$ iff $a > b$
   (iii) $a \rho b$ iff $a = b$ (iv) $a \rho b$ iff $a = b + 1$ and
   (v) $a \rho b$ iff $a + b \leq 4$.
5. If ρ is a relation on the set $\{1,2,3,4,5\}$, list the ordered pairs in ρ where
   (i) $a \rho b$ if 3 divides $a - b$, (ii) $a \rho b$ if $a + b = 6$,
   (iii) $a \rho b$ if $a - b$ is even, (iv) $a \rho b$ if lcm (a, b) is odd,
   (v) $a \rho b$ if $a^2 = b$.

6. If $\rho_1 = \{(1, 2), (2$
   $\rho_2 = \{(1, 1), (1, 2$
   from $\{1, 2, 3\}$ to
7. Determine wh
   symmetric, an
   (i) $a + b = 0$
   (iii) $a - b$ is a ra
   (v) $ab \geq 1$,
   (vii) $a = 1$,
8. For each of
   Reflexive, sy
   (i) $\rho \subseteq Z^+ \times$
   (ii) $\rho \subseteq Z \times Z$
   (iii) ρ is the r
   (iv) ρ is the
   (v) ρ is the
   perpend
9. Determine
   antisymm
   (i) a is tall
   (iii) a has t
   (v) a and
10. Which o
    Relation
    (i) $\{(2, 4)$
    (ii) $\{(1, 1)$
    (iii) $\{(2, 2)$
    (iv) $\{(1, 1)$
    (v) $\{(1, 3$
11. If $\{1, 2,$
    If $a + d$
12. Let A =
    Show t
13. Suppos
    equiva
    $|A|$.
14. Let F b
    if the
    equiva

6. If $\rho_1 = \{(1, 2), (2, 3), (3, 4)\}$ and $\rho_2 = \{(1, 1), (1, 2), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3), (3, 4)\}$ be the relations from $\{1, 2, 3\}$ to $\{1, 2, 3, 4\}$. Find (i) $\rho_1 \cup \rho_2$, (ii) $\rho_1 \cap \rho_2$, (iii) $\rho_1 - \rho_2$, (iv) $\rho_2 - \rho_1$.

7. Determine whether the relation ρ on the set of all real numbers is reflexive, symmetric, antisymmetric and/or transitive, where $(a, b) \in \rho$ iff.
   (i) $a + b = 0$ (ii) $a = \pm b$
   (iii) $a - b$ is a rational number, (iv) $a = 2b$
   (v) $ab \geq 1$, (vi) $ab = 0$
   (vii) $a = 1$, (viii) $a = 1$ or $b = 1$.

8. For each of the following relations, determine whether the relation is Reflexive, symmetric, antisymmetric and/or transitive:
   (i) $\rho \subseteq Z^+ \times Z^+$, where $a \rho b$ if a divides b.
   (ii) $\rho \subseteq Z \times Z$, where $a \rho b$ if a divides b.
   (iii) ρ is the relation on Z, where $a \rho b$ if $a + b$ is odd.
   (iv) ρ is the relation on Z, where $a \rho b$ if $a - b$ is even.
   (v) ρ is the relation on the set of lines in a plane such that $a \rho b$ if a is perpendicular to b.

9. Determine whether the relation ρ on the set of people is reflexive, Symmetric, antisymmetric and/or transitive, where $a \rho b$ if
   (i) a is taller than b, (ii) a and b were born on the same day,
   (iii) a has the same first name as b, (iv) a is a spouse of b,
   (v) a and b have a common grand parent.

10. Which of the following relations on the set $\{1, 2, 3, 4\}$ is/are equivalent Relations? Find the properties of an equivalent relation that the others lack.
   (i) $\{(2, 4), (4, 2)\}$
   (ii) $\{(1, 1), (2, 2), (3, 3), (4, 4)\}$
   (iii) $\{(2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4)\}$
   (iv) $\{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4)\}$
   (v) $\{(1, 3), (1, 4), (2, 3), (2, 4), (3, 1), (3, 4)\}$.

11. If $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and ρ be the relation defined by $(a, b), (c, d) \in \rho$ if $a + d = b + c$, prove that ρ is an equivalence relation.

12. Let $A = \{1, 2, 3, 4\}$ and $\rho = \{(1, 1), (1, 3), (2, 2), (2, 4), (3, 1), (3, 3), (4, 2), (4, 4)\}$. Show that ρ is an Equivalence Relation.

13. Suppose that ρ is an equivalence relation on a finite set A, and every equivalence class has the same cardinality m. Express $|\rho|$ in terms of m and $|A|$.

14. Let F be any partition of the set S. Define a relation on S by $x \rho y$ if and only if there is a set in F which contains both x and y. Then show that ρ is an equivalence relation and the equivalence classes of ρ are the sets of F.

15. (a) If ρ is the relation defined on Z such that $a \rho b$ if and only if $a^2 - b^2$ is divisible by 3, show that ρ is an equivalence relation.
(b) If ρ is the relation on N defined by $a\rho b$ if and only if a/b is a power 2, Show that ρ is an equivalence relation.

16. If ρ is the relation on A = {1, 2, 4, 6, 8} defined by $a\rho b$ if and only if a/b is an integer, show that ρ is a partial ordering on A.

17. (a) If ρ is the equivalence relation on the set A = {0, 1, 2, 3, 4} given by {(0, 0), (1, 1),(0, 4), (1, 3), (2, 2), (3, 1), (3, 3), (3, 6),(4, 4), (4, 0)}, find the distinct equivalence classes of ρ.
(b) If ρ is the equivalence relation on A = {1, 2, 3, 4, 5, 6 } given by {(1,1),(1,5),(2,2),(2,3),(2,6),(3,2),(3,3),(3,6),(4,4),(5,1),(5,5),(6,2),(6,3),(6,6)}, find the partition of A induced by ρ.

18. If ρ and S are equivalence relations on A = {a, b, c, d, e} given by ρ = {(a, a), (a, b), (b, a), (b, b), (c, c), (d, d), (d, e), (e, d), (e, e)} and S = {(a, a), (b, b), (c, c), (d, d), (e, e), (a, c), (c, a), (d, e), (e, d)}, determine the partitions A induced by (a) ρ^{-1}, (b) $\rho \cap$ S.

19. Draw the Hasse diagram representing the partial ordering P = {(a, b) : a divides b} on {1, 2, 3, 4, 5, 6}

20. Draw the Hasse diagram for the divisibility relation {2, 4, 5, 10, 12, 20, 25}.

21. Draw the Hasse diagram for the "less than or equal to" relation on {0, 2, 5, 10, 11, 15}.

B Multiple Choice Questions (MCQ)

1. **Relation R is defined on the set of all integers as a R b iff $a \neq b$. Then R is**
(a) transitive (b) symmetric
(c) reflexive (d) anti- symmetric

2. **R is defined on the set of all integers as a R b iff $ab \geq 1$. Then R is**
(a) symmetric and transitive (b) reflexive
(c) transitive only (d) equivalence

3. **Relation R is defined on Z (set of integers) as x R y iff $x \geq y^2$. Then R is**
(a) reflexive (b) anti- symmetric
(c) symmetric (d) equivalence

4. **R = {$(x, y) : |x + y| = 2, x, y \in \mathbb{Z}$}. Then R is**
(a) reflexive (b) symmetric
(c) transitive (d) anti-symmetric

5. **The relation {(a, b) : a, b real and $a \leq b$} is reflexive**
(a) True (b) False

6. **The relation {(1, 1**
reflexive
(a) True

7. **If R and S are tw**
(a) R ∪ S is an eq
(b) $R^{-1} \cup S^{-1}$ is a
(c) $R^{-1} \cap S^{-1}$ is a
(d) none of thes

8. **Number of pos**
(a) 2^6
(c) 2^{12}

9. **The ordered p**
(a) a R b iff $a \leq$
(c) a R b iff $a =$

10. **Relation R is**
(a) {(1, 1)}
(c) {(0, 1), (0,

11. **Relation R is**
iff the words
(a) reflexive,
(b) reflexive,
(c) reflexive,
(d) not reflex

12. **R be the rel**
the number
(a) 1
(c) 3

13. **On the set**
not prime
is not
(a) Reflexiv
(c) Transiti

14. **For any a,**
(a) An equi
(b) Symme
(c) Reflexiv
(d) Reflexiv

6. The relation $\{(1, 1), (1, 2), (2, 3), (3, 3)\}$ defined on the set $A = \{1, 2, 3\}$ is reflexive
(a) True (b) False

7. If R and S are two equivalence relations on a set A then
(a) $R \cup S$ is an equivalence relation
(b) $R^{-1} \cup S^{-1}$ is an equivalence relation
(c) $R^{-1} \cap S^{-1}$ is an equivalence relation
(d) none of these

8. Number of possible relations defined on the set with six elements is
(a) 2^6 (b) 2^{36}
(c) 2^{12} (d) $2^{36} - 1$

9. The ordered pair (3, 1) belongs to the relation R on $\{1, 2, 3\}$ defined as
(a) $a\,R\,b$ iff $a \leq b$ (b) $a\,R\,b$ iff $a \neq b$
(c) $a\,R\,b$ iff $a = b$ (d) $a\,R\,b$ iff $b = a + 2$

10. Relation R is on the set $\{1, 2, 3\}$ defined as $a\,R\,b$ iff $a + b \leq 1$
(a) $\{(1, 1)\}$ (b) $\{(1, 1), (3, 4)\}$
(c) $\{(0, 1), (0, 0)\}$ (d) none of these

11. Relation R is on the set of all words in English dictionary defined as $x\,R\,y$ iff the words x and y has atleast one letter in common. Then R is
(a) reflexive, but not symmetric and transitive
(b) reflexive, symmetric, but not transitive
(c) reflexive, but not symmetric and transitive
(d) not reflexive, symmetric and transitive

12. R be the relation defined on $\mathbb{Z}$ as $R = \{(a, b) : a - b \text{ is divisable by } 3\}$. Then the number of pair wise disjoint subsets in partion induced by R is
(a) 1 (b) 2
(c) 3 (d) infinitely many

13. On the set of integers $\mathbb{Z}$ defined a relation ρ by $(a, b) \in \rho$ iff a^2 and b^2 are not prime to each other. Then ρ is not an equivalence relation because ρ is not
(a) Reflexive (b) Symmetric
(c) Transitive (4) Antisymmetric

14. For any $a, b \in N$ define $a\,R\,b$ iff a divides b, then R is
(a) An equivalence relation
(b) Symmetric and transitive but not reflexive
(c) Reflexive but not symmetric and transitive
(d) Reflexive and transitive but not symmetric.

15. **Let R be a symmetric and transitive relations on a set A. Then**
(a) R is reflexive and hence an equivalence relation
(b) R is reflexive and hence partial order.
(c) R is reflexive and hence not an equivalence relation.
(d) None of the above.

16. **The number of equivalence relations on the set {1, 2, 3, 4} is**
(a) 15 (b) 16 (c) 24 (d) 4

17. **Suppose A is a finite set with n elements. The number of elements in the largest equivalence relation of A is**
(a) n (b) n^2 (c) 1 (d) $n+1$

18. **Let X and Y be two equivalence relation on a set. Consider the following assertions :**
(i) $X \cup Y$ is an equivalence relation (ii) $X \cap Y$ is an equivalence relation
Which one of the following is correct?
(a) Both assertions are true
(b) assertion (i) is true but (ii) is not true
(c) assertion (ii) is true but (i) is not true.
(d) neither (i) nor (ii) is true.

19. **The binary relation R = {(1,1), (2,1), (2,2), (2,3), (2,4), (3,1), (3,2), (3,3), (3,4)} on the set A = {1, 2, 3, 4} is**
(a) reflexive, symmetric and transitive
(b) neither reflexive nor irreflexive but transitive
(c) irreflexive, symmetric and transitive
(d) irreflexive and antisymmetric

20. **The number of binary relations on a set with n elements is**
(a) n^2 (b) $2n$
(c) 2^{n^2} (d) none of the above

21. **A relation R is defined on the set of integers as x R y iff $(x + y)$ is even. Which one of the following statements is TRUE?**
(a) R is not an equivalence relation
(b) R is an equivalence relation having 1 equivalence class.
(c) R is an equivalence relation having 2 equivalence classes.
(d) R is an equivalence relation having 3 equivalence classes.

22. **A relation R is defined on ordered pairs of integers as follows : $(x, y)R(u, v)$ if $x < u$ and $y > v$. Then R is**
(a) Neither a partial order nor a equivalence relation
(b) A partial order but not a total order
(c) A total order
(d) An equivalence relation

24. **Let S be a set of n elements. and smallest equivalence re**
(a) n and n
(c) n^2 and 0.

24. **Consider the binary relat $\{x, y, z\}$. Which one of the**
(a) R is symmetric but not
(b) R is not symmetric but
(c) R is both symmetric a
(d) R is neither symmetri

25. **The binary relation S = f**
(a) Neither reflexive nor
(c) Transitive and reflexi

26. **Let R and S be any two one of the following sta**
(a) $R \cap S$, $R \cup S$ are both
(b) $R \cup S$ is an equivala
(c) $R \cap S$ is an equivala
(d) Neither $R \cup S$ nor R

27. **Let R be the relation only if a and b are dist one of the following s**
(a) R is symmetric and
(b) R is reflexive but n
(c) R is transitive but
(d) R is symmetric bu

28. **A binary relation R o**
(a, b) R (c, d) iff $a \leq c$
P : R is reflexive
Q : R is transitive.
Which one of the fol
(a) Both P and Q are
(c) P is false and Q a

29. **Let $n(A) = m$ and n that can be defined**
(a) $m + n$

30. **The number of rela**
(a) 2^4

23. **Let S be a set of n elements. The number of ordered pairs in the largest and smallest equivalence relation on S are**
(a) n and n (b) n^2 and n
(c) n^2 and 0. (d) n and 1

24. **Consider the binary relation : R = {(x, y), (x, z), (z, x), (z, y)} on the set {x, y, z} . Which one of the following is TRUE?**
(a) R is symmetric but not antisymmetric
(b) R is not symmetric but antisymmetric
(c) R is both symmetric and antisymmetric
(d) R is neither symmetric nor antisymmetric.

25. **The binary relation S = f (empty set) on the set A = {1, 2, 3} is**
(a) Neither reflexive nor symmetric (b) Symmetric and reflexive
(c) Transitive and reflexive (d) Transitive and symmetric

26. **Let R and S be any two-equivalence relation on a nonempty set. Which one of the following statements is TRUE?**
(a) R ∩ S, R ∪ S are both equivalance relation.
(b) R ∪ S is an equivalance relation.
(c) R ∩ S is an equivalance relation
(d) Neither R ∪ S nor R ∩ S is an equivalance relatrion.

27. **Let R be the relation on the set of positive integers such that a R b if and only if a and b are distinct and have a common divisor other than 1. Which one of the following statements about R is TRUE?**
(a) R is symmetric and reflexive but not transitive.
(b) R is reflexive but not symmetric and not transitive.
(c) R is transitive but not reflexive and not symmetric.
(d) R is symmetric but not reflexive and not transitive.

28. **A binary relation R on N × N is defined as follow :**
(a, b) R (c, d) iff $a \leq c$ or $b \leq d$. Consider the following proposition :
P : R is reflexive
Q : R is transitive.
Which one of the following statements is TRUE?
(a) Both P and Q are true (b) P is true and Q are false
(c) P is false and Q are true (d) Both P and Q are false

29. **Let n(A) = m and n(B) = n. What will be the number of distinct relations that can be defined from A to B?**
(a) $m + n$ (b) mn (c) 2^{mn} (d) 2^{m+n}

30. **The number of relations defined on the set {1, 2, 3, 4} is**
(a) 2^4 (b) 2^{16} (c) 2^3 (d) 2^2

31. If $R = \{(x, y) : x, y \in Z \text{ and } 5|(x - y)$, then R is
(a) Not a relation
(b) An antisymmetric relation
(c) An equivalence relation
(d) A relation which is not reflexive.

32. Let R be a relation in the set of integers Z, defined by $a\,R\,b$ iff $(a - b)$ is even. Then R is
(a) Reflexive only
(b) Reflexive but symmetric only
(c) Symmetric and transitive only
(d) An equivalence relation.

33. An eqivalence relation is always symmetric
(a) true
(b) false

34. Which of the following relations is symmetric but neither reflexive nor transitive for a set A = {1, 2, 3}
(a) R = {(1, 2), (1, 3), (1, 4)}
(b) R = {(1, 2), (2, 1)}
(c) R = {(1, 1), (2, 2), (3, 3)}
(d) R = {(1, 1), (1, 2), (2, 3)}

35. Let R be a relation in the set N given by $R = \{(a, b) : a + b = 5, b > 1\}$. Which of the following will satisfy the given relation ?
(a) $(2, 3) \in R$
(b) $(4, 2) \in R$
(c) $(2, 1) \in R$
(d) $(5, 0) \in R$

36. Which of the following relations is reflexive but not transitive for the set T = {7, 8, 9}?
(a) R = {(7, 7), (8, 8), (9, 9)}
(b) R = {(7, 8), (8, 7), (8, 9)}
(c) = {0}
(d) R = {(7, 8), (8, 8), (8, 9)}

37. Let I be a set of all lines in XY plane and R be a relation in I defined as $R = \{(l_1, l_2) : l_1$ is parallel to $l_2\}$. What is the type of given relation?
(a) Reflexive relation
(b) Transitive relation
(c) Symmetric relation
(d) Equivalence relation

38. $(a, a) \in R$ implies that $(a, a) \in R$ for all $a, a \in A$. This condition is for which of the following relations?
(a) Equivalence relation
(b) Reflexive relation
(c) Symmetric relation
(d) Universal relation

39. Consider the relation R' (x, y) if and only if $x, y > 0$ over the set of non-zero rational numbers then R' is
(a) not equivalence relation
(b) an equivalence relation
(c) transitive and asymmetric relation
(d) reflexive and antisymmetric relation

40. If S is relation from a non-empty set A to a non-empty set B then
(a) $S = A \cup B$
(b) $S = A \cap B$
(c) $S \subseteq A \times B$
(d) $S = A \times B$

41. If A = {a, b, c}. Which of the follow
(a) {(a, a), (c, c), (a, b)}
(c) {(b, b), (c, c), (c, b)}

42. A relation R on a set of real numb
(a) reflexive
(c) Transitive

A. Short and Long Answer Type Que

1. {(1, 3), (1, 5), (3, 3), (3, 5)}; Dom
3. $R^{-1} = \{(4, 1), (4, 2), (5, 1), (5, 3)\}$;
4. (i) {(1, 1), (1, 2), (1, 3), (2, 2), (2,
(ii) {(2,1), (3, 1), (3, 2)}
(iii) {(1, 1), (2, 2), (3, 3)}
(iv) (2, 1), (3, 2)
(v) {(1, 1), (1, 2), (1, 3), (2, 1),
5. (i) {(1, 1), (1, 4), (2, 2), (2, 5),
(ii) {(1, 5), (2, 4), (3, 3), (4, 2),
(iii) {(1, 1), (1, 3), (1, 5), (2, 4),
(iv) {(1, 1), (1, 3), (1, 5), (3, 1
(v) {(1, 1), (2, 4)}.
6. (a) $\rho_1 - \rho_2 = \rho_2$ (b) $\rho_1 \cap$
(d) $\rho_2 - \rho_1 = \{(1, 1), (2, 1),$
7. (i) symmetric, (ii) reflexi
(iii) reflexive, symmetric a
(v) reflexive and symmetr
(vii) reflexive and transiti
8. (i) reflexive, antisymmetr
(iv) reflexive, symmetric
9. (i) transitive; (ii) reflexi
(iii) reflexive, symmetric
(v) reflexive and symme
10. (i) neither reflexive not
(ii) equivalence relatio
(iii) neither reflexive n
(iv) equivalence relatio
(v) lacks all the three

41. If $A = \{a, b, c\}$. Which of the following relation of A is reflexive?
(a) $\{(a, a), (c, c), (a, b)\}$ (b) $\{(a, a), (a, b), (b, c), (b, b), (c, c)\}$
(c) $\{(b, b), (c, c), (c, b)\}$ (d) none of these

42. A relation R on a set of real numbers defined as $R = \{(a, b) : a \leq b^2\}$ is
(a) reflexive (b) Symmetric
(c) Transitive (d) none of these

Answers

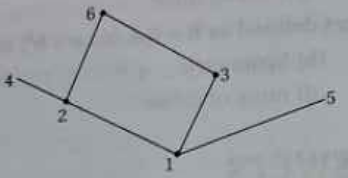
A. Short and Long Answer Type Questions

1. $\{(1, 3), (1, 5), (3, 3), (3, 5)\}$; $\text{Dom}(R) = \{1, 3\}$; $\text{Ran}(R) = \{3, 5\}$.
3. $R^{-1} = \{(4, 1), (4, 2), (5, 1), (5, 3)\}$; $\overline{R} = \{(2, 5), (3, 4)\}$.
4. (i) $\{(1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 3)\}$
(ii) $\{(2,1), (3, 1), (3, 2)\}$
(iii) $\{(1, 1), (2, 2), (3, 3)\}$
(iv) $(2, 1), (3, 2)$
(v) $\{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (3, 1)\}$.
5. (i) $\{(1, 1), (1, 4), (2, 2), (2, 5), (3, 3)\}$
(ii) $\{(1, 5), (2, 4), (3, 3), (4, 2), (5, 1)\}$
(iii) $\{(1, 1), (1, 3), (1, 5), (2, 4), (3, 1), (3, 3), (3, 5), (4, 2), (4, 4), (5, 1), (5, 3), (5, 5)\}$
(iv) $\{(1, 1), (1, 3), (1, 5), (3, 1), (3, 3), (3, 5), (5, 1), (5, 3), (5, 5)\}$
(v) $\{(1, 1), (2, 4)\}$.
6. (a) $\rho_1 - \rho_2 = \rho_2$ (b) $\rho_1 \cap \rho_2 = \rho_1$ (c) $\rho_1 - \rho_2 = \varnothing$
(d) $\rho_2 - \rho_1 = \{(1, 1), (2, 1), (2, 2), (3, 1), (3, 2), (3, 3)\}$.
7. (i) symmetric, (ii) reflexive, symmetric and transitive;
(iii) reflexive, symmetric and transitive; (iv) none;
(v) reflexive and symmetric; (vi) symmetric;
(vii) reflexive and transitive; (viii) symmetric.
8. (i) reflexive, antisymmetric and transitive; (ii) transitive; (iii) symmetric;
(iv) reflexive, symmetric and transitive; (v) symmetric.
9. (i) transitive; (ii) reflexive, symmetric and transitive;
(iii) reflexive, symmetric and transitive; (iv) symmetric;
(v) reflexive and symmetric.
10. (i) neither reflexive not transitive;
(ii) equivalence relation(also partial ordering)
(iii) neither reflexive nor symmetric;
(iv) equivalence relation;
(v) lacks all the three properties.

17. (a) $[\{0, 4\}, \{1, 3\}, \{2\}]$; (b) $[\{1, 5\}, \{2, 3, 6\}, \{4\}]$

18. (a) $[\{a, b\}, \{c\}, \{d, e\}]$ (b) $[\{a\}, \{b\}, \{c\}, \{d, e\}]$.

19.



B. Multiple Choice Questions (MCQ)

1. b	2. a	3. b	4. b	5. a	6. b	7. c	8. b	9. d	10. d
11. b	12. c	13. a	14. d	15. d	16. a	17. b	18. c	19. b	20. c
21. c	22. a	23. b	24. d	25. d	26. c	27. d	28. a	29. c	30. c
31. c	32. d	33. a	34. b	35. a	36. a	37. d	38. c	39. b	40. c
41. b	42. d								

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